Mathematics Contest

FIRST ROUND

For Colorado Students Grades 7 - 12

November 4, 2000

- The lattice point (m,n) is a point in the plane where both m and n are integers.
- The symbol n!, read as n factorial, means $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$, the product of the first n positive integers.
- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b and c are the side lengths of a right triangle with hypotenuse c.

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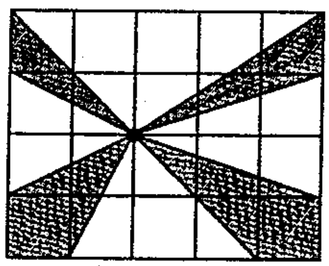
1. A lecture hall has 50 rows of seats. There are 12 seats in the first row, 14 seats in the second row, and so on, with two more seats in each row than in the previous row. How many seats are there in the lecture hall?

2. Determine the area of the shaded region in the 4 by 5 grid made up of 20 1 by 1 squares.



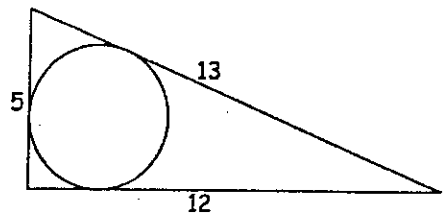
3. Determine the quantity $a+b+c$ given that $\frac{a}{4-a} = \frac{b}{5-b} = \frac{c}{7-c} = 3$.

4. What number is directly above 223 in this array?

		1		
	2	3	4	
5	6	7	8	9
10 11	12	13	...	

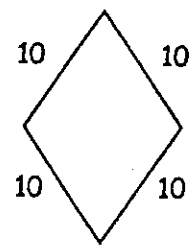
5. Determine <u>all</u> positive integers that make this statement true:
$n^2 + (n+1)^2 + (n+2)^2 = (n+3)^2 + (n+4)^2$.

6. Determine the radius of the circle inscribed in a 5–12–13 right triangle.



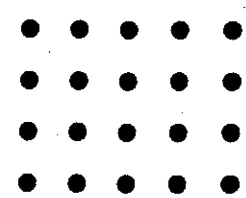
7. Factor $n^4 - 20n^2 + 4$ into polynomials with integer coefficients.

8. Find the area of a rhombus that has a side of length of 10 and diagonals whose lengths differ by 4.



9. Determine the <u>number</u> of positive integral solutions to $\frac{1}{m} + \frac{1}{n} = \frac{1}{2^3 \cdot 3^2}$. In other words, <u>how many</u> ordered pairs (m, n), with m and n both integers and $m \geq n$, are there that satisfy this equation?

10. Two points are randomly and simultaneously selected from the 4 by 5 grid of 20 lattice points $\{(m,n): 1 \leq m \leq 5 \text{ and } 1 \leq n \leq 4\}$.
Determine the probability that the distance between the two points is an integer.



11. 35! has 41 digits when written out and can be printed as a diamond read row by row. What is the missing center digit?

1
0 3 3
3 1 4 7 9
6 6 3 8 6 1 4
4 9 2 9 □ 6 6 6 5
1 3 3 7 5 2 3
2 0 0 0 0
0 0 0
0

November 2000

1. 3050; The total is $(12+14+16+\ldots+110)=\frac{122(50)}{2}=3050$

2. 7; Each shaded area is found by subtracting areas of triangles from rectangles.

3. a + b+ c=12; a=12-3a, b=15-3b, c=21-3c. $4(a+b+c)=48$.

4. 195; each row ends in a perfect square; search for 225 and back up.

5. n=10; simplifying gives $n^2-8n-20=0$ with roots n=10, -2.

6. r=2; This 5-12-13 triangle can be broken down into 3 smaller ones each with vertex at the center of the circle. Then $\frac{1}{2}(r)(12)+\frac{1}{2}(r)(13)+\frac{1}{2}(r)(5)=\frac{1}{2}(5)(12)=30$ or 15r=30, so that r=2, d=4.

7. $(n^2-4n-2)(n^2+4n-2)$; $n^4-20n^2+4=n^4-4n^2+4-16n^2=(n^2-2)^2-(4n)^2=(n^2-4n-2)(n^2+4n-2)$

8. 96; subdivide the rhombus into 4 right triangles each with side lengths $\frac{a}{2}$ and $\frac{(a+4)}{2}$. Then $100=\left(\frac{a}{2}\right)^2+\left(\frac{a+4}{2}\right)^2$ and $400=a^2+(a+4)^2=a^2+a^2+8a+16=2a^2+8a+16$. Finally $200=a^2+4a+8$ or $a^2+4a-192=0=(a-12)(a+16)$. If a=12, area=96.

9. 18; $72(m+n)=mn$ gives $n=\frac{72m}{m-72}=\frac{72m-(72)(72)+(72)(72)}{m-72}=72\frac{m-72}{m-72}+\frac{(72)(72)}{m-72}=72+\frac{72^2}{m-72}$. For n to be an integer m-72 must be one of the $(7)(5)=35$ divisors of $72^2=(2^6)(3^4)$. There are 17 solutions with $m>n$ and 1 more when m=n=144.

10. $\frac{72}{190}$; There are $\binom{20}{2}=190$ total ways of selecting 2 of the 20 points. Now count favorable events. You can take any 2 of 5 on 4 different horizontal lines for a total of $\binom{5}{2}4=40$ or take any 2 of 4 on 5 vertical lines for a total of $\binom{4}{2}5=30$. Finally there are two 3-4-5 right triangles.

11. 6; cast out nines (carefully).

Mathematics Contest

FINAL ROUND

For Colorado Students Grades 7 - 12

February 3, 2001

- The lattice point (m,n) is a point in the plane where both m and n are integers.
- The symbol n!, read as n factorial, means $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$, the product of the first n positive integers.
- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b and c are the side lengths of a right triangle with hypotenuse c.

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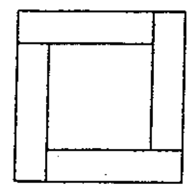
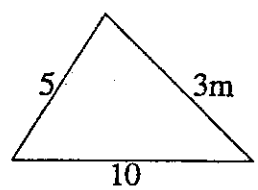
1. A square is surrounded by four congruent rectangles. If each of the rectangles has perimeter of 18 units, what is the total sum of the areas of the four rectangles and interior square?



2. Determine the three integer values of m so that 5, 10, and 3m are the measures of the sides of a triangle.



3. Determine the number of positive integers that have their digits in strictly increasing order. Include single digit numbers. Here are four examples: 148, 3, 123456, 19.

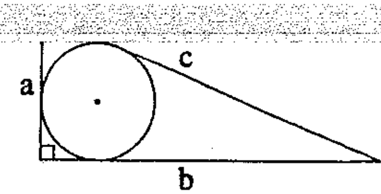
4. Determine all 3-digit numbers abc that equal the sum of the factorials of their digits. That is: $100a + 10b + c = a! + b! + c!$ (Note that a must be at least 1 in order for abc to be a 3-digit number.)

5. Solve the following system of equations:

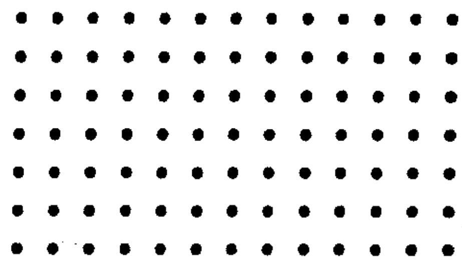
$$\frac{2}{x-1} + \frac{5}{y-2} = 1$$

$$\frac{1}{x-1} + \frac{3}{y-2} = 2$$

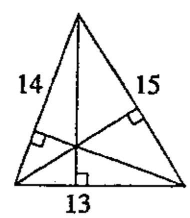
[illegible] triangle ABC with sides shown where c is the length of the hypotenuse.



7. Two points are randomly and simultaneously (without replacement) selected from the 7 by 13 grid of lattice points $\{(m,n): 1 \le m \le 13,\ 1 \le n \le 7\}$. Determine the probability that the distance between them is an integer.



8. A triangle has sides of length 13, 14, 15. The altitude to one side is an integer. Determine its length.



9. Factor each of the following into a product of polynomials each with <u>integer</u> coefficients. (You are <u>not</u> asked to find the roots.). For example, $x^3+1=(x+1)(x^2-x+1)$ is such a factorization.

   (a) $x^9+x^6+x^3+1$
   (b) $x^{12}+x^9+x^6+x^3+1$ (Hint: Multiply this geometric sum by x^3-1.)
   (c) Generalize

10. Determine the <u>number</u> of positive integral solutions to $\frac{1}{m}+\frac{1}{n}=\frac{1}{2^{50}\cdot 3^{40}}$. In other words, <u>how many</u> ordered pairs (m,n) with m and n both integers and $m \ge n \ge 1$, are there that satisfy this equation?

11. Determine all polynomials p(x) such that (x-1)p(x+1) = (x+2)p(x) and p(2) =12. (Hint: try carefully selected values for x.)

SOLUTIONS TO FINAL ROUND
FEBRUARY 2001

1. 81; Label the sides of a small rectangle with a, b. Then 2a + 2b = 18, a + b = 9. Total area is $(a+b)^2 = 81$.

2. m = 2, 3,4; Trial and error will do it. Or solve $5 + 3m > 10, 15 > 3m$.

3. $2^9 - 1$; There are 9 single digit type, $\binom{9}{2}$ two-digit type, $\binom{9}{2}$ ~~two-digit type~~, $\binom{9}{3}$ three-digit type and so on. The total is $\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\ldots+\binom{9}{9} = 2^9 - 1$. Alternatively choose <u>any</u> subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ except the empty set. Each subset corresponds to an increasing number.

4. x = 6/7, y = 7/3; Let A = 1/x-1, B = 1/y-2. Then solve 2A + 5B = 1, A + 3B = 2 for A = -7 and B = 3; Finally, -7 = 1/x-1, and 3 = 1/y-2 yielding x = 6/7 and g= 7/3.

5. 145; Approach this by trial and error. The digit a must be in $\{1,2,3,4,5\}$ since 6! = 720 is too big. Now start with a = 1 and try values for b and c.

6. a + b – c or 2ab/a +b + c; Draw segments from each vertex of the triangle to the center of the circle. Equating the sum of the areas of the three triangles formed to the area of the original triangle gives $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}ab$. Now solve for r. Algebraic manipulation will give the simpler form a + b – c.

7. $\frac{965}{4095} \approx 24\%$; Integer lengths occur horizontally and vertically $7\binom{13}{2}+13\binom{7}{2}$ times.
[Alternatively: There are $7 \cdot 13 = 91$ points. Each can be paired with the other 6 in that column or 12 in that row for a total of $91(6+12) = 1638$. Now divide by 2 to remove duplications: 819]. The integer 5 occurs as the hypotenuse of a 3 – 4 – 5 triangle $2 \cdot 9 \cdot 4 + 3 \cdot 10 \cdot 2$ times. The integer 10 occurs in a 6 – 8 – 10 triangle 10 times, while 13 occurs in 5 –12 – 13 triangles 4 times. There are a total of $\binom{91}{2} = 4095$ possible pairs of points.

8. The altitude to side 14 is $h = 12$; Let h be the altitude to side 14.
$x^2 + h^2 = 13^2$ and $(14-x)^2 + h^2 = 15^2$. Subtract, and
$196 - 28x + x^2 - x^2 = 225 - 169$. Then x = 5, h = 12.

9. (a) $(x^3+1)(x^6+1)$; Factor by grouping: $x^6(x^3+1)+(x^3+1)=(x^3+1)(x^6+1)$

(b) $(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)$; $\frac{x^{15}-1}{x^3-1}=$

$\frac{(x^5-1)}{(x-1)}\frac{(x^{10}+x^5+1)}{(x^2+x+1)}$. Now use long division to complete.

10. 4091 ; Let $A=2^{50}\cdot 3^{40}$. $A(m+n)=mn$, $m=\frac{An}{n-A}=\frac{An-A^2+A^2}{n-A}=A+\frac{A^2}{n-A}$. m will be an integer whenever $n-A$ is a divisor of $A^2=2^{100}\cdot 3^{80}$; there are $(101)(81)$ divisors of $2^{100}\cdot 3^{80}$. The number of ordered pairs (m,n) with $m>n$ is $[(101)(81)-1]\div 2+1$ after extracting out the one solution $m=n=2\cdot 2^{50}\cdot 3^{40}$ and dividing by 2 to account for symmetry.

11. $p(x)=2x^3-2x^2$; since $p(1)=0$, $x-1$ is a factor. Now let $x=-2$, $-3p(-1)=0$ shows that $x+1$ is a factor. Now try $x=0$. $-p(1)=2p(0)$ gives $p(0)=0$ since $p(1)=0$. Hence x is a factor. Now, $p(x)=x(x-1)(x-2)q(x)$. Substitute this into the original equation:

$$(x-1)(x+1)x\,(x+2)\,q\,(x+1)=(x+2)\,x\,(x-1)(x+1)\,q\,(x)$$

Cancel; $q(x+1)=q(x)$ implies that $q(x)=\text{constant}$. Since $p(2)=12, q(2)=2$. Finally $p(x)=2x(x-1)(x+1)$.