Mathematics Contest

FIRST ROUND

For Colorado Students Grades 7 – 12

November 3, 2001

- The positive integers are 1, 2, 3, 4, 5, 6, 7, ... and so on.
- The symbol n!, read as n factorial, means $1 \cdot 2 \cdot 3 \cdots n$, the product of the first n positive integers.
- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b and c are the side lengths of a right triangle with hypotenuse c.

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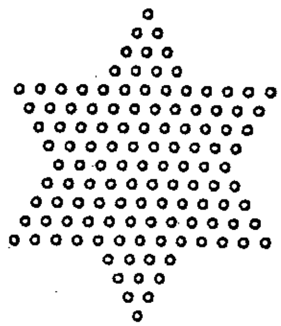
1. How many marbles does it take to fill an entire Chinese Checker board, placing one marble in each slot?



2. The lengths of two sides of an isosceles triangle are 18 and 41. Compute the area of the triangle.

3. Determine the positive integer b such that $\frac{b}{4}+\frac{4}{b}$ is simultaneously greater than 4 and less than 4.1.

4. Split 2001 into two numbers so that one is a multiple of 17 and the other one is a multiple of 103. In other words, find numbers a and b so that a + b = 2001 where a is a multiple of 17 and b is a multiple of 103.

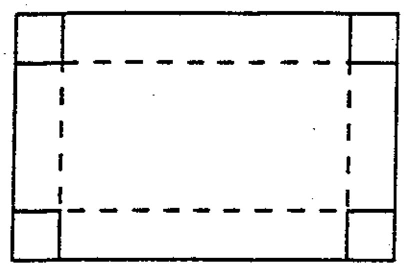
5. There are 6 = 3! permutations of 1, 2, 3.
   Treat each of these permutations as a 3-digit number.
   (a) Compute the sum of these 6 numbers.
   (b) Repeat with 1, 2, 3, 4. That is, determine the sum of all the permutations of 1, 2, 3, 4 (as shown to the right).

$$\begin{array}{r} 1234 \\ 1243 \\ 1324 \\ 1342 \\ \vdots \\ +4321 \\ \hline \end{array}$$

over

$$\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{599^2}\right)=\frac{n}{599}.$$

7. An open rectangular box can be formed by cutting identical squares off the four corners of a rectangular piece of cardboard and folding up the four sections that stick out. For a particular piece of cardboard, the <u>same</u> volume results whether squares of size 1 or squares of size 2 have been cut out. Compute the resulting volume if squares of size 3 are cut out.



8. To the right is a system of 25 equations in 26 unknowns.
   (a) What is the value of $A+B+C+D+\cdots+X+Y+Z$?
   (b) What is the value of $A+Z$?

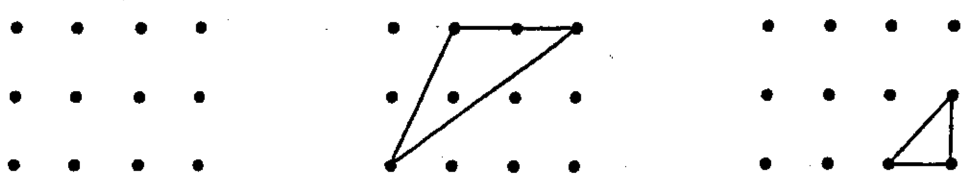
$$\begin{aligned} A+B &= 1\\ B+C &= 2\\ C+D &= 3\\ D+E &= 4\\ &\vdots\\ X+Y &= 24\\ Y+Z &= 25 \end{aligned}$$

9. In order to list the integers from 0 through 7 in binary notation $\{0,1,10,11,100,101,110,111\}$ we must write a total of twelve 1's. Compute the total number of 1's needed to list, in binary notation, all the integers

   (a) from 0 through 15.

   (b) from 0 through 31.

10. How many different triangles (with positive area) are there whose 3 vertices are among the 12 points in the 3 by 4 diagram? Two such triangles are drawn as examples.



11. What is the remainder when $x^{33}+x^{22}+x^{11}+2x+3$ is divided by $1+x+x^2$?

Solutions **First Round** **November 3, 2001**

1. 121; $1+2+3+\cdots+13=91$ on the very large triangle plus $3(1+2+3+4)=30$ for the three small ones left over.

2. 360; the two equal sides must be 41 making the altitude 40 since $41^2=40^2+9^2$.

3. 15; One procedure is trial and error. Try b= 1, 2, ...

4. (a) = $765=17(45)$ and $b=1236=103(12)$; From $17m+103n=2001$ write $17m = 2001 - 103n$ and start taking multiples of 103 away from 2001 until you hit a multiple of 17.

5. (a) 1332; Add the six numbers up. Or think about 222 being the "average" of the six numbers and $6 \times 222 = 1332$.
   (b) 66,6660; each column has 6 1's, 6 2's, 6 3's, and 6 4's. The first column sums to $6[1+2+3+4]=60$, the second to 600, the third to 6000 and the left most column sums to 60,000.

6. 450; The product equals $\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{6}\right)\left(1+\frac{1}{6}\right)\cdots\left(1-\frac{1}{599}\right)\left(1+\frac{1}{599}\right)$
$= \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{7}{6}\cdots \cdot\left(1+\frac{1}{599}\right)=\frac{3}{4}\left(1+\frac{1}{599}\right)$ after cancellation.
$=\frac{3}{4}\left(\frac{600}{599}\right)=\frac{450}{599}$

7. 24; Set $V(1)=(x-2)(y-2)$ and $V(2)=2(x-4)(y-4)$ equal to each other, giving $xy-2x-2y+4=2xy-8y-8x+32$ or $0=xy-6x-6y+28$.
   then $V(3)=3(x-6)(y-6)=3(xy-6x-6y+36)=3(-28+36)=24$

8. (a) 169; Adding every other equation gives $(A+B)+(C+D)+\cdots+(Y+Z)=$ $1+3+5+\cdots+25=13^2=169$.
   (b) 13; $A+2(B+C+\cdots+Y)+Z=1+2+3+\cdots=325$, $A+Z+325=2\cdot 169=338$.

Over.

9. (a) 32; Each 4 digit binary "word" looks like_ _ _ _. There are 2^4 of these requiring $4 \cdot 2^4$ digits. But half are 0's, half are 1's

UNIVERSITY OF NORTHERN COLORADO

MATHEMATICS CONTEST

FINAL ROUND

For Colorado Students Grades 7-12

February 16, 2002

- $1+2+3+\cdots+n = n(n+1)/2$.
- The binomial coefficient $\binom{n}{k}$ gives the number of ways of choosing k objects out of n objects.
- The symbol $n!$, read as n factorial, means $1 \cdot 2 \cdot 3 \cdots n$, the product of the first n positive integers.

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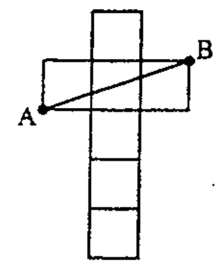
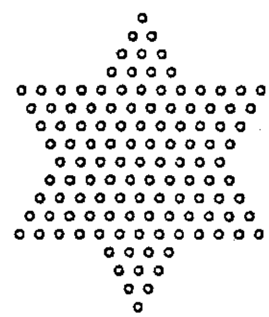
1. A cross is made up of seven congruent squares. If $\overline{AB}=20$, determine the area of the cross.

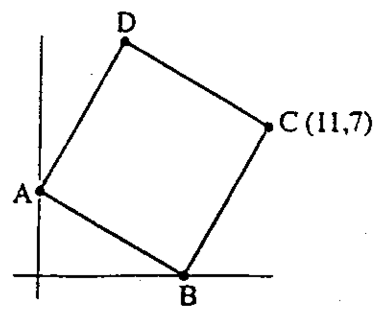


2. On the FIRST ROUND of this contest you were asked to determine the number of marbles it would take to fill a standard Chinese Checker board where each of the six "tails" were equilateral triangles with side length 4. How many marbles would it take if each of the six "tails" had side length

   (a) 5 (b) 6 (c) n ?



3. The square ABCD is positioned on the axes as shown. Point C has coordinates (11,7). Compute the area of the square.



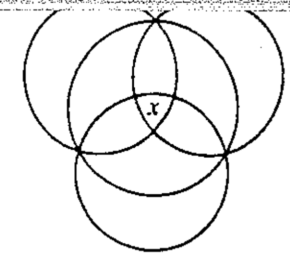
4. A 9 by 9 by 9 wooden cube is formed by gluing together $9^3 = 729$ small unit cubes. What is the greatest number of unit cubes that can be seen from a single viewpoint? (Here, you "see" a unit cube if you can see any part of it).

5. In order to list the integers from 0 through 7 in binary or base 2 notation we must write a total of 12 1's. Compute the total number of 1's needed to list in base 2 all the integers
   (a) from 0 through 63.
   (b) from 0 through $2^n - 1$.

0 = 0
1 = 1
10 = 2
11 = 3
100 = 4
101 = 5
110 = 6
111 = 7

over

6. Four overlapping circles enclose ten regions, as shown. Each region is to be filled with one of the integers 1 to 10, using each integer exactly once. The sum of the integers inside each of the circles is the same. Which integer is in the region marked by x ?



7. There are 4!=24 permutations of the integers 1, 2, 3, 4. Treat each of these permutations as a 4-digit number.
   (a) Compute the sum of these 24 numbers.
   (b) Repeat with 1, 2, 3, 4, 5. That is, determine the sum of all the permutations of 1, 2, 3, 4, 5.
   (c) Repeat with 1, 2, 3, ..., n.

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 1234
 1243
 1324
 1342
   ⋮
+4321
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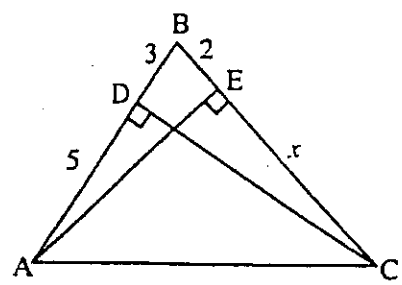
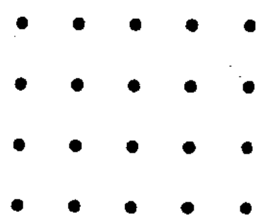
8. Let ABC denote a triangle all of whose angles are acute. Two altitudes are drawn, resulting in $\overline{AD} = 5$, $\overline{DB} = 3$, $\overline{BE} = 2$. Compute x, the length of $\overline{EC}$.



9. The symbol P_k stands for an integer whose base ten representation consists of k ones. For example, $P_3 = 111$, $P_5 = 11{,}111$. When P_6 is divided by P_2 the quotient $\frac{P_6}{P_2}$ is an integer whose base ten representation is a sequence containing only ones and zeros. In fact $\frac{P_6}{P_2} = \frac{111{,}111}{11} = 10{,}101$.
   (a) Determine the number of zeros in P_{12}/P_3.
   (b) Determine the number of zeros in P_{24}/P_4.

10. How many triangles (with positive area) are there whose 3 vertices are among the 20 points in the 4 by 5 diagram?



SOLUTIONS – FINAL ROUND
FEBRUARY 16, 2002

1. 280; Let x be length of side of a small square.
   $x + (3x)^2 = 400 = 10x^2; x^2 = 40 \;\; 7x^2 = 280$

2. (a) 181; $3(1+2+3+4+5)+(1+2+\ldots+16) = 3(15)+136 = 181$

   (b) 253; $3(1+2+3+4+5+6)+(1+2+\ldots+19) = 63+190 = 253$

   (c) $6n^2+6n+1$; $3(1+2+\ldots+n)+(1+2+\ldots+3n+1) = 6n^2+6n+1$

   Alternatively, subdivide into 12 smaller triangles to get $12[n(n+1)/2]+1 = 6n^2+6n+1$

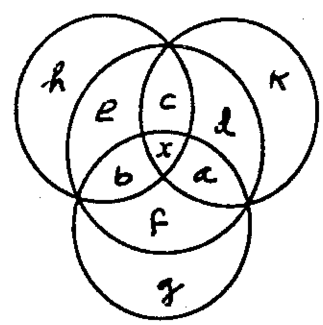
3. 65; $\overline{BC}^2 = 4^2+7^2 = 65$

4. 217; $3\cdot 8^2+3\cdot 8+1$. Ignoring the three edges, there are 8×8 cubes showing on each of 3 faces. Add those on the 3 edges, plus the corner one. Also notice
   $3\cdot 8^2+3\cdot 8+1 = 9^3-8^3 = (8+1)^3-8^3 = 8^3+3\cdot 8^2+3\cdot 8+1-8^3$. This "removes" the inner $8\times 8\times 8$ cube.

5. (a) 192; There are $64 = 2^6$ binary "words" of length 6 requiring $6\cdot 2^6$ digits. Half of these digits are 0's, half are 1's. So there are $6\cdot 2^5 = 192$ 1's present.

   (b) $n\cdot 2^{n-1}$

6. $x = 7$; $x+a+c+d+k = x+b+c+e+h$
   $= x+a+b+c+d+e+f$
   $k = e+b+f \qquad = x+a+b+f+g$
   $h = a+d+f$
   $g = c+d+e$, $g+h+k = 2(d+e+f)+a+b+c$
   $g+h+k$ is at most $8+9+10 = 27$ and
   RHS is at least $2[1+2+3]+4+5+6 = 27$
   so $\{g, h, k\} = \{8, 9, 10\}$, $\{a, b, c\} = \{4, 5, 6\}$ and $\{d, e, f\} = \{1, 2, 3\}$
   and finally, $x = 7$.



7. 66,660; Each column has 6 1's, 6 2's, 6 3's and 6 4's. The first column sums to $6[1+2+3+4] = 60$, the second to 600, the third to 6000 and the final left most column sums to 60,000

   (b) 3,999,960; Each column has 24 1's, 24' 2s,…24 5's. The first column sums to

$24[1+2+3+4+5] = 360$ The final sum is

$360[1+10+100+1{,}000+10{,}000] = 360[11111] = 3{,}999{,}960$

Alternatively, think of 33333 being the "average" of the 120 numbers.

$33333 \times 120 = 3{,}999{,}960$. For part (a) you could think of 2.5 2.5 2.5 2.5 as being the "average". So half are like 2222 and half like 3333. $12(2222) + 12(3333) = 66{,}660$

(c) $\frac{(n+1)!}{2}[1+10+10^2+\ldots+10^{n-1}] = \frac{(n+1)!}{2}\left[\frac{10^n-1}{9}\right]$.

8. $x = 10$; $AE^2 = 60, 9 + DC^2 = (x+2)^2, 25 + DC^2 = AC^2, 60 + x^2 = AC^2,$

$DC^2 = x^2 + 35, 9 + x^2 + 35 = (x+2)^2$ and $x = 10$

Or, using similar triangles $3/2 = (x+2)/8$.

9. (a) 6 zeros; $P_{12}/P_3 = 111,111,111,111/111 = 999{,}999{,}999{,}999/999 = (10^{12}-1)/10^3-1$

$= 10^9 + 10^6 + 10^3 + 1 = 1{,}001{,}001{,}001$.

(b) 15 zeros; $P_{24}/P_4 = (10^{24}-1)/10^4-1 = 10^{20}+10^{16}+10^{12}+10^8+10^4+1$ which has 6 1's and $21-6 = 15$ zeros.

In general, If n divides m P_m/P_n has $\left(\frac{m}{n}-1\right)(n-1)$ zeros.

10. 1056; $\binom{10}{3} - 4\binom{5}{3} - 5\binom{4}{3} - 2\binom{3}{3} - 2\binom{4}{3} - 2\binom{4}{3} - 2\binom{3}{3} - 4 = 1140 - 84 = 1056$