UNIVERSITY OF NORTHERN COLORADO

Mathematics Contest

FIRST ROUND

For all Colorado Students Grades 7 - 12

November 9, 2002

- The arithmetic mean of a set of n numbers is their sum divided by n.
- The median of a set of numbers is the middle number by size.
- The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- The nonnegative integers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 ...

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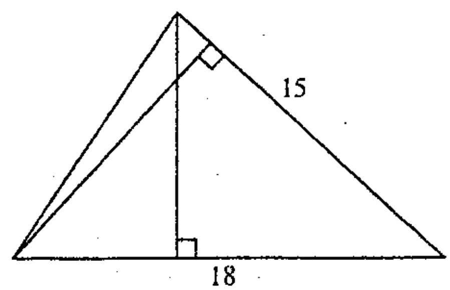
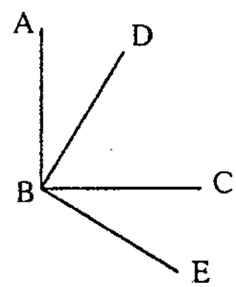
1. Two sides of a triangle have length 15 and 18.
The altitude to the side of length 18 is 10.
What is the length of the altitude to the side of length 15?

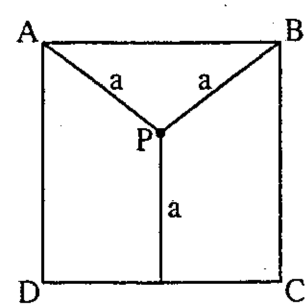


2. In the diagram $\angle ABC$ and $\angle DBE$ are right angles.
Let $m\angle ABD = x$, $m\angle DBC = x + y$ and $m\angle CBE = 2y$.
Find x.



3. There is just one integer value of n for which the median and the arithmetic mean of the five numbers 15, 13, 17, 27, and n are equal. Find n.

4. ABCD is a square with side length 16. Point P is an interior point whose distances to the point A and to the point B and to the side CD are equal. Find that distance a.



5. What is the smallest multiple of 75 that consists of just 1's and 0's?

over

6. A three digit number (100 through 999) is equal to the product of two numbers; one consists of a single digit, the other of two digits. The six digits are all different and are in the set $\{1, 2, 3, 4, 5, 6\}$. Determine a, b, c, d, e, f so that $a \cdot bc = def$.

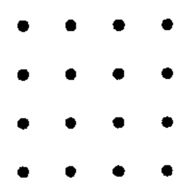
7. Determine all nonnegative integer values of n for which $\frac{n}{15-n}$ is a perfect square.

8. If a, b, c are different numbers so that

$$a^3 + 2002a + 13 = 0$$
$$b^3 + 2002b + 13 = 0$$
$$c^3 + 2002c + 13 = 0 \quad ,$$

determine the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

9. How many squares can you make using four of these sixteen dots as vertices? In any row or column, the dots are equally spaced as on a geoboard.



10. Let G(m) denote the largest power of 2 that divides m. For example, G(24)=3 since 2^3 is the largest power of 2 that divides 24. Complete the chart.

m	2	4	6	8	10	12	14	16	18	20	22	24
G(m)												3

11. In how many positive integers less than one million do each of the digits 1, 2, 3, 4, 5 appear exactly once, in any order? Here are two examples: 329,145 and 15,234.

BRIEF SOLUTIONS – FIRST ROUND
NOVEMBER 2002

1. 12; Area of the triangle is $\frac{1}{2}(18)(10)=90$ one way and $\frac{1}{2}(15)a$, another way. Set these equal and $90=\frac{1}{2}(15)a$, $a=12$.

2. 36°; $2x+y=90$, $x+3y=90$ gives $y=18$, $x=36$.

3. 3; set $(15+13+17+27+n)/5=(72+n)/5$ equal to each of 13, 15, 17, 27, n and n must be 3. The average is 15 and the mean is 15.

4. 10; Let x be the altitude from P to AB. Solve $a+x=16$, $a^2=x^2+64$.

5. 11100; The smallest multiple of $3\cdot 5^2$ has to be a multiple of 3 and an even multiple of 25. So it must end in two 0's and have 3 1's (casting out 3's).

6. $3(54)=162$; The only possible products are $2(\_\_3)$ and $3(\_\_4)$. Only $3(54)$ works.

7. $n=0, 3, 12$; Trial and error with $n=0, 1, 2, \ldots, 14$. (Assuming that n is an integer.)

8. -154; a, b, c are roots of $x^3+2002x+13=0$. Then $a+b+c=0$, $ab+ac+bc=2002$, $abc=-13$,

   $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+ac+bc}{abc}=\frac{2002}{-13}=-154$. Alternatively: Replace x by $\frac{1}{x}$ and find the sum of the roots. You get $\frac{1}{x^3}+\frac{2002}{x}+13=0$, and $1+2002x^2+13x^3=0$. The sum of the roots is $-\frac{2002}{13}=-154$.

9. 20; There are 9 small, 4 medium, 1 large and $6=4+2$ tilted.

10.

m	2	4	6	8	10	12	14	16	18	20	22	24
G(m)	1	2	1	3	1	2	1	4	1	2	1	3

11. 3600; There are 6 places to insert one of the 5 digits 0, 6, 7, 8, 9. For each of these 30 possibilities there are $5!=120$ ways to have 1, 2, 3, 4, 5 appear. Total = $30(120)$.

UNIVERSITY OF NORTHERN COLORADO

MATHEMATICS CONTEST

FINAL ROUND

For Colorado Students Grades 7-12

February 8, 2003

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, ...
- The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- A median of a triangle is a segment from a vertex to the midpoint of the opposite side.
- A lattice point (m,n) is a point in the plane with m and n being integers.

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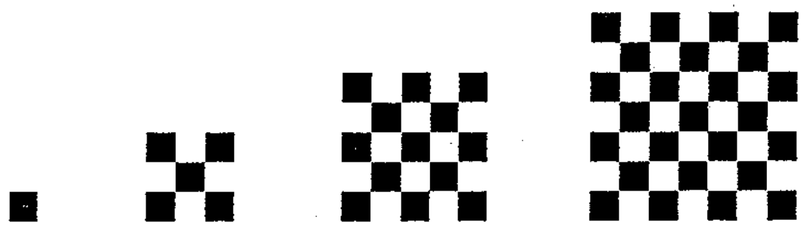
1. How many black tiles will be required to build the 10^{th} figure in this pattern?

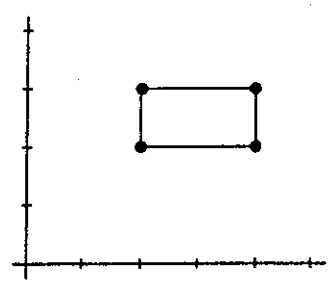


2. The number 19 appears in the 3^{rd} row and 4^{th} column. In which row and column does 2003 appear?

1	3	6	10	15	21	···
2	5	9	14	20	···	
4	8	13	19	···		
7	12	18	···			
11	17	···				
16	···					
⋮						

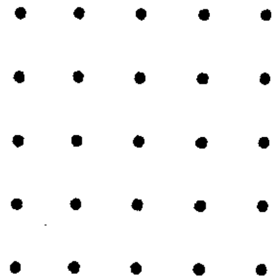
3. An isosceles triangle has one of its three medians equal to 15 inches and one of its three altitudes equal to 24 inches. There are exactly two different triangles for which this can be true.
   (a) Draw both triangles.
   (b) Compute the area of each.

4. Find the slope of the line that passes through the origin and bisects the area of the rectangle (having vertices at (2,2), (4,2), (4,3), and (2,3)) shown.



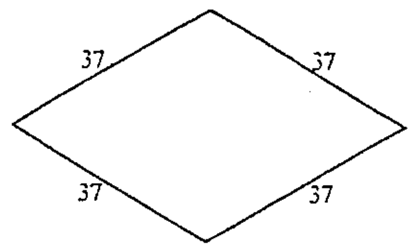
over

5. How many squares can you make using four of the twenty-five dots as vertices? In any row or column, the dots are equally spaced as on a geoboard.



6. What is the smallest positive multiple of 225 that consists of just 1's and 0's? Explain your reasoning. If you use a calculator for assistance, you must still explain why your answer is the smallest.

7. The area of a rhombus having side lengths 37 inches is 840 square inches. Determine the lengths of the two diagonals.



8. A four-digit integer (1000 through 9999) is equal to the product of three integers; one of these integers consists of a single digit, one is a two-digit integer and the other is a three-digit integer. The ten digits forming these four integers are all different. Find either one of the two possible four-digit integers. [On the first round you determined that 162=3·54, a similar factoring of a three-digit integer. Here you need to determine the ten values so that abcd=efg·hi·j using each digit 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once and only once.]

9. The number r is a fixed point of the function f if f(r)=r.
   (a) Determine a fixed point of $f(x)=x^3+3x^2+4x+1$.
   (b) Suppose $f(x)=x^3+6x^2+ax+b$ has exactly one fixed point. Determine a and b.

10. Let G(m) denote the largest power of 2 that divides m. For example, G(24)=3 since 2^3 divides 24 and 2^4 does not. On the first round you collected data for G(m).
    (a) Using data for G(m), conjecture a formula or algorithm that could be used to evaluate G(m).
    (b) Explain your reasoning.

11. The 35 lines $L_1, L_2, L_3, \ldots, L_{35}$ lie in a plane. The lines $L_5, L_{10}, L_{15}, \ldots, L_{35}$ are parallel to each other while $L_1, L_6, L_{11}, L_{16}, \ldots, L_{31}$ all pass through a given point A.
    (a) What is the maximum number of points of intersection of pairs of lines using the complete set of 35 lines?
    (b) Generalize the problem, and solve your generalization.

BRIEF SOLUTIONS-FINAL ROUND

FEBRUARY 8, 2003

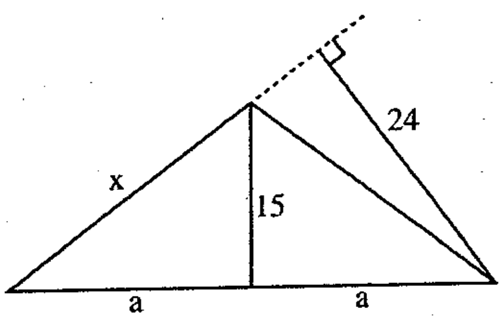
1. 181; There will be $9^2 + 10^2$ black tiles.

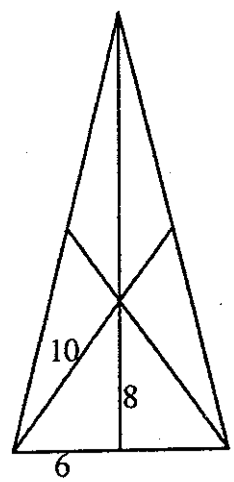
2. 14^{th} row, 50^{th} column; The triangular numbers across the first row are given by $\frac{n(n+1)}{2}$.

   Guess at n so that $n(n+1)$ is close to 4006.

3. 300 and 144; For one triangle, Area = $\frac{1}{2}(2a)15 = 15a = \frac{1}{2}x(24) = 12x$. Then $x = \frac{5}{4}a$.

   Now use $x^2 = a^2 + 225$ and a = 20, giving Area = 300. For the other one, Area = $\frac{1}{2}(12)(24) = 144$, since altitudes and medians all intersect two-thirds of the way from vertices.





4. $\frac{5}{6}$; The line must pass through the center (3, 2 ½) of the rectangle.

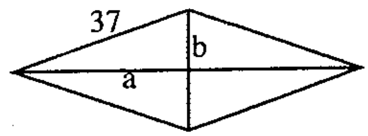
5. 50; Count all sizes including "tilted" ones.

6. 11,111,111,100; Since $225 = 5^2(3^2)$, the desired multiple must be divisible by 9 and, therefore, must have nine 1's (or 18 1's or …) present. Multiples of 25 end in 25, 50, 75, or 00.

7. 24 and 70; $\frac{1}{2}ab = 210, ab = 420, (a+b)^2 = a^2 + b^2 + 2ab = 37^2 + 840 = 47^2$;

$a + b = 47, a(47-a) = 420$. Now solve $a^2 - 47a + 420 = 0$.

Factoring, $(a-12)(a-35) = 0$ gives a = 12, 35 and b = 35, 12.



8. 8596 and 8970; Trial and error will produce 8596 = 2 x 14 x 307 and 8970 = 1 x 26 x 345.

9. (a) r = -1; $r^3 + 3r^2 + 4r + 1 = r$ implies $(r+1)^3 = 0$ and r = -1

(b) a = 13, b = 8; $x^3 + 6x^2 + ax + b = x$ implies $x^3 + 6x^2 + (a-1)x + 6 = 0$.

If there is exactly one fixed point r, this latter cubic must be

$(x-r)^3 = x^3 + 3x^2r + 3xr^2 - r^3$. Equating coefficients gives $-3r = 6$, $3r^2 = a - 1$ and

$b = -r^3$ or r = -2, a = 13 and b = 8.

10. G(m) is the number of rightmost zeros in the binary expansion of m. If m = 24, then, in base 2, m = 11000 and G(24) = 3. Or, divide m by 2 until you reach an odd number. G(m) = # of divisions.

11. 554; The 7 parallel lines intersects the 7 that pass though a point A in 49 points; the remaining 21 lines can intersect these 14 lines in 21(14) points. Finally, the 21 lines intersect themselves in $\binom{21}{2}$ points. Including the point A, the total is

$$49 + 21(14) + \binom{21}{2} + 1.$$