UNIVERSITY OF NORTHERN COLORADO

Mathematics Contest

FIRST ROUND

For all Colorado Students Grades 7-12

November 1, 2003

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, . . .
- The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- A triangle is isosceles if exactly two sides have the same length.

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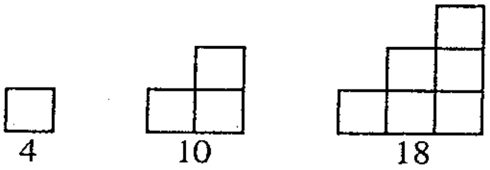
1. What is the smallest positive integer that 1260 should be multiplied by so that the product produced is a
   (a) perfect square? (b) perfect cube?

2. Find a positive integer m such that both $4m^2$ and $\left(4m^3\right)/3$ are 4-digit integers. (Hint: There is only one.)

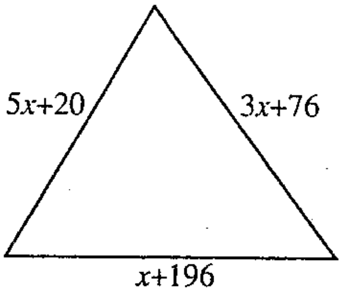
3. The following staircase patterns of squares require 4, 10 and 18 toothpicks to make:



   How many toothpicks are needed to make the 10^{th} pattern in the sequence?

4. The product of three consecutive even numbers is the 8-digit number 87xxxxx8. Find these three numbers and supply the five missing digits.

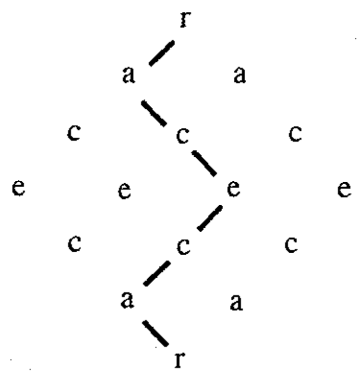
5. What is the greatest perimeter possible given that the triangle is isosceles? [Note: You do not know which two sides are equal].



Over

6. In a hat, place 8 slips of paper numbered 1, 2, 3, 4, 5, 6, 7, 8. Draw three slips out, one at a time without replacement. What is the probability that you draw the numbers in increasing order.

7. How many different paths of length 6 from the top *r* to the bottom *r* are there that spell *racecar*? One such path is drawn as an example.



8. Consider the pattern to the right:

(a) Express each of the 2^{nd} and 3rd lines as a perfect square of a positive integer, as shown in the first line.

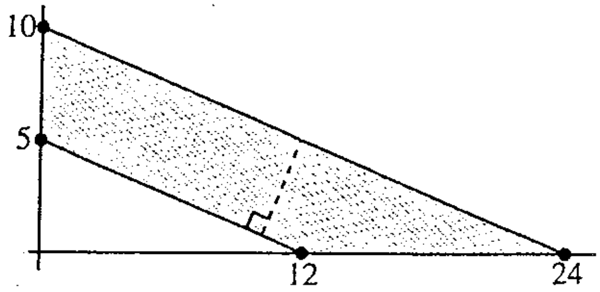
$1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25 = 5^2$

$2 \cdot 3 \cdot 4 \cdot 5 + 1 =$

$3 \cdot 4 \cdot 5 \cdot 6 + 1 =$

(b) Write out the next two lines of this pattern; include expressions as perfect squares.

9. The shaded area in the first quadrant is that of a trapezoid. Determine the height (indicated by the dotted line) of this trapezoid.



10. A sequence of positive integers begins with the two terms $a_1 = 13$ and $a_2 = 21$. Each subsequent term is the average of <u>all</u> the preceding terms. Compute the value of a_{2003}, the 2003^{rd} term in the sequence.

11. In a certain code, each of the 26 letters of the alphabet is represented by the number that is its usual alphabetic order. In other words, a = 1, b = 2, c = 3, ..., z = 26. An English word is encoded by <u>multiplying</u> the numbers that represent its letters. What English word (not gibberish) is encoded by the number 120,175? As an example, the number 10 encodes the word "be" (here, $10 = 2 \cdot 5 = 5 \cdot 2 = 1 \cdot 10 = 10 \cdot 1$ and "eb", "aj", and "ja" are not words).

BRIEF SOLUTIONS TO FIRST ROUND
NOVEMBER 2003

1. (a) 35 (b) 7350; Since $1260 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7$, you need an extra 5 and 7 to make a perfect square, and an extra 2, 3 and two each of the 5 and 7 to make a perfect cube.

2. m=18; Trial and error, $1000 \leq 4m^2 \leq 9999$, and the fact that m must be a multiple of 3 yields the result.

3. 130; By direct count, the pattern starts as 4, 10, 18, 28, 40… . Dividing by 2 gives 2, 5, 9, 14, 20, …, a sequence with successive differences of 3, 4, 5, 6… . Continuing with this pattern of differences you get 2, 5, 9, 14, 20, 27, 35, 44, 54, 65 and then doubling the 65 gives 130.

4. 442, 444, 446 and 5, 2, 6, 6, 0; The cube root of the given eight digit number is "about" 443. Trial and error will produce the three factors.

5. 832; If $5x + 20 = 3x + 76$, then $x = 28$ and the triangle has sides 160, 160 and 224. If $5x + 20 = x + 196$, $x = 44$ and the sides are 240, 240, and 208. If $3x + 76 = x + 196$, $x = 60$ and the sides are 256, 256, 320 yielding the maximum perimeter.

6. $1/6$; For any three numbers there are 3! = 6 orders possible; but only one is in increasing order.

7. 20; Each path of length 6 must have 3 "rights" and 3 "lefts". Choose the 3 "rights" in $\binom{6}{3} = 20$ ways.

8. 11^2, 19^2, $4 \cdot 5 \cdot 6 \cdot 7 \cdot + 1 = 29^2$ and $5 \cdot 6 \cdot 7 \cdot 8 + 1 = 41^2$.

9. $60/13$; Viewing 13 as the base of the small triangle, the area of the small triangle is $30 = \frac{1}{2}(13)h$ and $h = 60/13$. With 26 as the base of the large triangle, the area of the large triangle is $120 = \frac{1}{2}(26)k$ and $k = 120/13$. Then $k - h = 120/13 - 60/13 = 60/13$ is the height of the trapezoid.

10. 17; The sequence produced is 13, 21, 17, 17, 17… .

11. WEEKS; $120{,}175 = 5 \cdot 5 \cdot 11 \cdot 19 \cdot 23$ with e = 5, k = 11, s = 19, w =23. $25 = 5 \cdot 5$ is problematic. ASK AWAY = $1 \cdot 19 \cdot 11 \cdot 1 \cdot 23 \cdot 1 \cdot 25$ is ruled out as we requested a single word. Here $a = 1$.

UNIVERSITY OF NORTHERN COLORADO
MATHEMATICS CONTEST
FINAL ROUND
For Colorado Students Grades 7-12
February 7, 2004

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, … .
- The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

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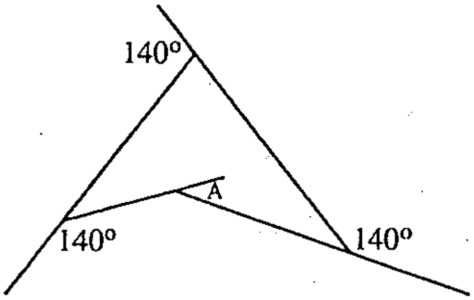
1. Determine the measure of angle A in the figure.



2. If $P = 3^{2004} + 3^{-2004}$ and $Q = 3^{2004} - 3^{-2004}$ compute the numerical value of $P^2 - Q^2$.

3. Conjecture the n-th line of the pattern to the right and give a proof.

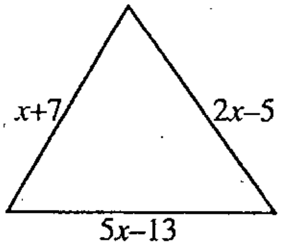
$$1 \cdot 2 \cdot 3 \cdot 4 + 1 = 5^2$$

$$2 \cdot 3 \cdot 4 \cdot 5 + 1 = 11^2$$

$$3 \cdot 4 \cdot 5 \cdot 6 + 1 = 19^2$$

$$\cdots$$

4. Determine all integer values of x so that $x + 7$, $2x - 5$ and $5x - 13$ are the measures of the sides of a triangle.



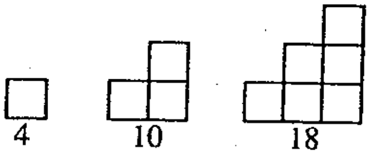
Over

5. Let $A = 2^2 + 4^2 + 6^2 + \cdots + 1000^2$ and $B = 1^2 + 3^2 + 5^2 + \cdots + 999^2$.
Express $A - B$ as an integer.

6. Determine integers a, b and c (which may be positive or negative but none are zero) such that a and b are the roots of $ax^2 + bx + c = 0$.

7. Show, by factoring, that $n^4 + 4^n$ is a composite (i.e., not a prime) number for $n > 1$. For example, when $n = 3$, $3^4 + 4^3 = 81 + 64 = 145$, and 145 can be factored as $5 \cdot 29$.

8. The following staircase patterns of squares require 4, 10 and 18 toothpicks to make.



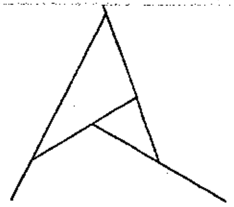
Let $T(n)$ denote the number of toothpicks required to make the n th figure in the sequence. Conjecture and prove a formula for $T(n)$. So far, we have $T(1) = 4$, $T(2) = 10$, and $T(3) = 18$.

9. (a) Express $1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 3 + 2 + 1$ as a polynomial in n.

(b) Express $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 + (n-1)^2 + \cdots + 3^2 + 2^2 + 1^2$ as a polynomial in n.

(c) Express $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 + (n-1)^2 + \cdots + 3^2 + 2^2 + 1^2$ as a linear combination of binomial coefficients.

10. Determine those positive integers n that have $n/2$ integral divisors. Prove your claim. Include 1 and the integer n itself when counting divisors. For example, the integer $n = 10$ is <u>not</u> an example since 10 has 4 divisors (1, 2, 5, and 10) and not $10/2 = 5$ divisors.



Solutions to the Final Round of the UNC Math Contest (February 2004)

1. *Sixty degrees.*

Fill in the supplementary angles and extend one line.

2. *Four.*

Use the identity $P^2 - Q^2 = (P+Q)(P-Q)$ with $P = 3^{2004} + 3^{-2004}$ and $Q = 3^{2004} - 3^{-2004}$ to deduce that $P^2 - Q^2 = (2 \cdot 3^{2004})(2 \cdot 3^{-2004}) = 4$.

3. $n(n+1)(n+2)(n+3) + 1 = [n(n+3)+1]^2$

Clearly $p(n) = n(n+1)(n+2)(n+3) + 1 = n(n^3 + 6n^2 + 11n + 6) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1$ is a fourth-degree polynomial in n. From the data, $p(n)$ must be the square of a function $f(n)$ that takes positive integral values $f(1) = 5, f(2) = 11, f(3) = 19, \ldots$. If $f(n)$ is a polynomial $a + bn + cn^2 + \ldots$, it's degree must be 2, since it's square $p(n)$ has degree 4. Now we must solve for the coefficients of $f(n)$, using the identity $p(n) = n^4 + 6n^3 + 11n^2 + 6n + 1 = [f(n)]^2 = [cn^2 + bn + a]^2 = c^2n^4 + \ldots + a^2$. Note that $f(n)$ is only determined up to a ± sign. We seek the solution that is positive when n is large. By comparing coefficients of powers of n, we see that $c^2 = 1$ and $a^2 = 1$. Thus by taking $c = 1$, we find that $f(n) = n^2 + bn + a$ where $a = \pm 1$. Now use the data $f(1) = 5 \implies 1 + b + a = 5$ and $f(2) = 11 \implies 4 + 2n + a = 11$ to solve for $b = 3$ and $a = +1$.

Or, one could easily guess the form $(n^2 + an + b)^2$, deduce that $b = 1$ and try a few values of a, arriving at $a = 3$ quickly since 5 is the result when $n = 1$.

Alternatively, we can determine $f(n)$ from the condition that $n(n+1)(n+2)(n+3) = p(n) - 1 = f(n)^2 - 1 = [f(n)+1][f(n)-1]$. must be a product of factors whose difference is 2. This can be arranged by regrouping $n(n+1)(n+2)(n+3) = [n(n+3)][(n+1)(n+2)]$. A solution is found by taking $f(n) - 1 = n(n+3) = n^2 + 3n$, $f(n) + 1 = (n+1)(n+2) = n^2 + 3n + 2$, whence $f(n) = n^2 + 3n + 1$.

4.$x = 5, 6, 7.$

Using the triangle inequality three times, we obtain (i) $(x+7) + (2x-5) > 5x - 13$, hence $x < 15/2$; (ii) $(2x-5) + (5x-13) > x + 7$, hence $x > 25/6 > 4$; (iii) $(x+7) = (5x-13) > 2x - 5$, hence $x > 1/4$, which is less restrictive than (ii). Thus the only possible candidates are $x = 5, 6, 7$. After checking that all three candidates give positive values for all sides of the triangle, we retain all three.

5. $A - B = 500,500$.

$$A - B = (2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + \ldots + (1000^2 - 999^2)$$

$$= (2-1)(1+2) + (4-3)(3+4) + (6-5)(5+6) + \ldots + (1000-999)(999+1000)$$

$$= (1 + 2 + 3 + 4 + \ldots + 999 + 1000) = (1000)(1001)/2 = 500,500.$$

6. $a = -2, b = 4, c = 16.$

Since $ax^2 + bx + c = a(x-a)(x-b) = a[x^2 - (a+b)x + ab]$ we have $b = -a(a+b)$. Solving for $b = \frac{-a^2}{(a+1)} = -[\frac{a^2-1+1}{a+1}] = -[(a-1) + \frac{1}{a+1}]$ Since a and b are required to be integers, $\frac{1}{a+1}$ must be an integer. The only integer a that allows $\frac{1}{a+1}$ to be an integer is $a = -2$ or $a = 0$. Since $a = 0$ is excluded in the statement of the problem, $a = -2$. Then $b = 4$. Finally, $c = a^2 b = 16$.

7. When n is even, $n^4 + 4^n$ clearly has 2 as a factor. When n is odd,

$$n^4 + 4^n = (n^2)^2 + (2^n)^2 = (n^2 + 2^n)^2 - 2n^2 2^n = A^2 - B^2 = (A + B)(A - B)$$

where $A = n^2 + 2^n, B = n2^{\frac{n+1}{2}}; A + B = (n^2 + 2^n + n2^{\frac{n+1}{2}}), A - B = (n^2 + 2^n - n2^{\frac{n+1}{2}})$. Since n is odd, the exponent of the term $2^{\frac{n+1}{2}}$ is indeed an integer. Finally we need to check that the second factor $A - B$ isn't 1, (so that the factorization is nontrivial). Rewrite the second factor, regarded as a quadratic in n, by completing the square to obtain $A - B = (n - 2^{\frac{n-1}{2}})^2 + (2^n - 2^{n-1}) \geq 2^n - 2^{n-1} = 2^{n-1} > 1$, for $n > 1$.

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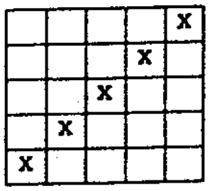
8. $T(n) = n(n + 3)$.

From the picture below, $2T(4)$ = all toothpicks in the big square, except 4 toothpicks located at two corners. In the big square, there are in total 5×6 vertical toothpicks and an equal number of horizontal ones. Thus $2T(4) = 2 \times (5 \times 6) - 4$, hence $T(4) = 5 \times 6 - 2$. In general, by the same reasoning, $T(n) = (n + 1)(n + 2) - 2 = n^2 + 3n$.

An alternate solution is based on the difference equation $T(n) - T(n - 1) = 2(n + 1)$, suggesting that $T(n)$ is a quadratic function $an^2 + bn + c$. The 3 unknown coefficients in the quadratic can be determined from the first 3 known values: $T(1) = a + b + c = 4, T(2) = 4a + 2b + c = 10, T(3) = 9a + 3b + c = 18$.

A third solution: Each triangular figure contains $1 + 2 + \ldots + n$ horizontal toothpicks, plus n more at the base. Now rotate the triangular figure 90°. By symmetry the vertical pieces have the same count. Then

$$T(n) = 2(1 + 2 + \ldots + n + n) = 2\frac{n(n+1)}{2} + 2n = n^2 + 3n.$$



(9. A) n^2 because $\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$.
(9.B) $\frac{(2n^3+n)}{3}$ because $\frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n(2n-1)}{6} = \frac{n}{6}[2n^2+3n+1+2n^2-3n+1] = \frac{n}{6}[4n^2+2] = \frac{2n^3+n}{3}$
(9.C) The $k = 3$ diagonal of the Pascal triangle begins with $1, 4, 10, 20, 35, \ldots$. The sequence of sums of two consecutive terms is $1, 5, 14, 30, 55, \ldots$,which is exactly the cumulative sum of squares: $1^2, 1^2 + 2^2, 1^2 + 2^2 + 3^2, \ldots$. By direct computation or by induction,$1^2 + 2^2 + \ldots + n^2 = \binom{n+1}{3} + \binom{n+2}{3}$. Then $1^2 + 2^2 + \ldots + n^2 + (n-1)^2 + (n-2)^2 + \ldots + 2^2 + 1^2 = \binom{n+1}{3} + \binom{n+2}{3} + \binom{n}{3} + \binom{n+1}{3} = \binom{n}{3} + 2\binom{n+1}{3} + \binom{n+2}{3}$.

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10. The only solutions are 8 and 12.

In general the number of divisors $d(n)$of n satisfies $2\sqrt{n} \geq d(n)$, since each factor of n less than $\sqrt{n}$ can be paired with exactly one complementary factor greater than $\sqrt{n}$. Thus in this problem $2\sqrt{n} \geq d(n) = \frac{n}{2}$ gives $4 \geq \sqrt{n}$, hence $16 \geq n$. Now trial and error can be used to determine which numbers $16 \geq n$ actually satisfy the conditions stated in the problem.