UNIVERSITY OF NORTHERN COLORADO

Mathematics Contest

FIRST ROUND

For all Colorado Students Grades 7-12

November 6, 2004

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, … .
- The sum of the first n positive integers is $1+2+3+\cdots+n=n(n+1)/2$.
- The arithmetic mean of a set of n numbers is their sum divided by n.

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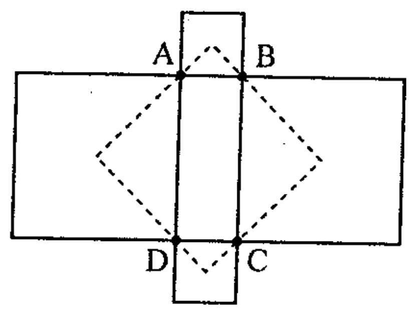
1. Find unequal positive integers m and n so that $\frac{1}{m}+\frac{1}{n}=\frac{1}{5}$.

2. Place a square on each side of rectangle ABCD. Connect the centers of each of these four squares forming the quadrilateral shown with the dotted segments. Find the area of this quadrilateral, if the dimensions of rectangle ABCD are 3 and 8.



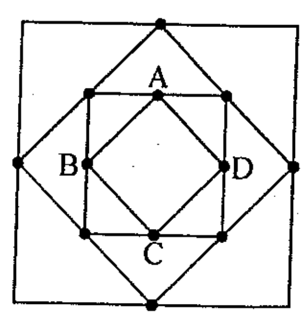
3. Consider the three integers 2718, 3875, and 4486. Find the largest integer n which yields the same remainder when each of these three integers is divided by n. As an example, the largest integer n for the three integers 12, 17, and 37 is $n=5$.

4. The arithmetic mean of a set of 50 numbers is 32. The arithmetic mean of a second set of 70 numbers is 53. What is the arithmetic mean of the sets combined?

5. Consider a square having side length 40 with three successive inscribed squares formed by connecting midpoints of sides of previous squares.

   (a) What is the perimeter of the smaller square ABCD?

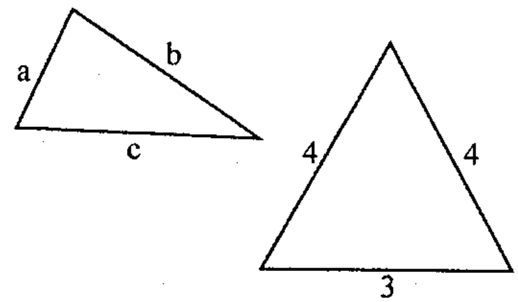
   (b) What is the area of square ABCD?



Over

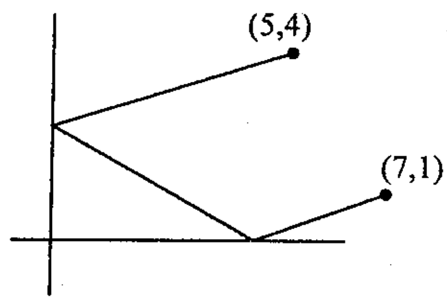
6. A farmer purchased 100 farm animals for a total of \$4500. Calves cost \$120 each, goats \$50 each, and lambs \$25 each. If the farmer obtained at least one animal of each type, how many of each type did the farmer purchase?

7. The three sides of a triangle have positive integer lengths a, b, and c satisfying the property $a \le b \le c$. For example, $3 \le 4 \le 4$ would give the triangle drawn to the right. Find the number of different (noncongruent) triangles satisfying this property for the case $c = 9$. Caution: 2, 4, and 9 cannot be the lengths of the three sides even though $2 \le 4 \le 9$.



8. What is the 2004th number in the following sequence: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, ...? As an example, the 10th number is 4.

9. Connect the points (5, 4) and (7, 1) by three line segments which meet the x-axis and y-axis as in the picture. What is the length of the shortest such path joining these points?



10. In the number spiral, give the next <u>three</u> numbers to the right of the numbers 11, 2, 1, 6, 19.

13	14	15	16	17			
12	3	4	5	18			
11	2	1	6	19	☐	☐	☐
10	9	8	7	20			
		...	22	21			

BRIEF SOLUTIONS TO FIRST ROUND
NOVEMBER 2004

1. $m = 6,\ n = 30$; First trial and error: since integers below 5 will not work, try $m = 6$ and conclude that n = 30. Alternatively from $5(n+m) = mn$, solve for $m = \frac{5n}{n-5} = 5 + \frac{25}{n-5}$. Only $n = 6$ will give unequal n and m.

2. Area = 60.5 ; The rectangle has area 24. The four triangles have area $\frac{1}{2}(8\cdot 4)+\frac{1}{2}(8\cdot 4)+\frac{1}{2}\left(3\cdot\frac{3}{2}\right)+\frac{1}{2}\left(3\cdot\frac{3}{2}\right)=36.5$

3. 13 ; $2718 = an + r$, $3875 = bn + r$, $4486 = cn + r$. Then n must divide $3875 - 2718 = 1157 = 13\cdot 89$; also n must divide $4468 - 2718 = 1768 = 13\cdot 176$ and n must divide $4486 - 3875 = 611 = 13\cdot 47$.

4. 44.25 ; The sum of the 50 numbers is $50\cdot 32$; The sum of the 70 numbers is $70\cdot 53$. The average of the 120 numbers is $(50\cdot 32 + 70\cdot 53)/120$.

5. (a) $40\sqrt{2}$ (b) 200 ; The successive side lengths of the four squares are 40, $20\sqrt{2}$, 20, and $10\sqrt{2}$. Hence the perimeter of the smallest square is $4\cdot 10\sqrt{2}$ and its area is $(10\sqrt{2})^2 = 200$.

6. There are four answers ; from $c + g + l = 100$ and $24c + 10g + 5l = 900$ replace l with $100 - c - g$, to obtain $19c + 5g = 400$. Solve for g as $g = \frac{400 - 19c}{5} = 80 - \frac{19}{5}c$. Try $c = 5$ to get $g = 61$, $l = 34$. Then try $c = 10$, $c = 15$, $c = 20$, yielding the other three answers: (10, 42, 48), (15, 23, 62) and (20, 4, 76).

7. 25 ; Try counting by systematic listing of all cases.

8. 51; $n(n+1)/2$ is a triangular number that gives the total number of entries in the sequence up through a subsequence 1, 2, . . , n ending in n. The value $n = 62$ yields a total of 1953 entries. Hence the 51st entry in the subsequence 1, 2, 3, . . . , 62 is the 2004th number.

9. 13 ; First reflect the segment containing (7, 1) across the x-axis. Then reflect the new (longer) segment across the y-axis. The distance between (5, 4) and (-7, -1) is 13.

10. 40, 69, 106 ; Look for a pattern.

UNIVERSITY OF NORTHERN COLORADO

FINAL ROUND

For Colorado Students Grades 7-12

January 29, 2005

- The odd positive integers are 1, 3, 5, 7, 9, 11 ...
- The absolute value of x, denoted $|x|$, has $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x \leq 0$.
- A regular polygon is a polygon with all sides having equal length, and all interior angles equal in measure.
- A geometric sequence consists of terms $a, ar, ar^2, ar^3 \ldots$
- A rational number is a ratio a/b of two integers, with $b \neq 0$.

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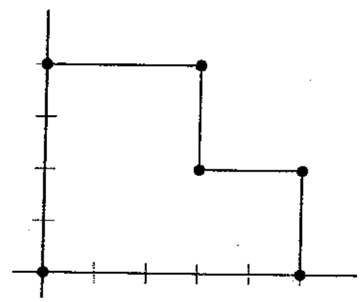
1. Find three different odd positive integers a, b and c so that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{3}$$

2. The diagram shows an L-shaped region in the first quadrant of the xy plane. The six vertices shown have coordinates (0,0), (0,4), (3,4), (3,2), (5,2) and (5,0). Determine the slope of the line through the origin that divides the area of this region exactly in half.

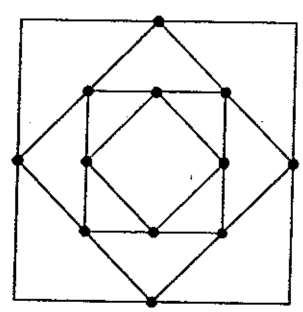


3. The number 3 can be expressed as a sum of one or more positive integers in four ways: 3, 1+2, 2+1, and 1+1+1. Note that 1+2 and 2+1 are considered as two different ways. In how many ways:
(a) can 4 be so expressed?
(b) can 100 be so expressed?
(c) can n be so expressed? Give a proof, and explain your reasoning.

4. There are a total of n students in a middle school. Four elevenths of n are in the 7th grade, a few sevenths of n are in the 8th grade and the remaining 324 students are in grade 9. Determine the total enrollment n.

5. A square with side length 40 has successive inscribed squares formed by connecting midpoints of sides of previous squares.

(a) Give the geometric sequence that describes the perimeters of the squares (starting with 160) as they shrink. List the first six terms.
(b) When would the perimeter of any inscribed square be a rational number?



Over

6. What is the 2005th number in the following sequence:

1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, ...?

7. The three sides of a triangle have integer lengths a, b and c satisfying the inequality $a \le b \le c$.
(a) How many noncongruent triangles satisfy this condition for $c = 10$?
(b) Determine a formula for the number of noncongruent triangles for general c. Explain your reasoning.

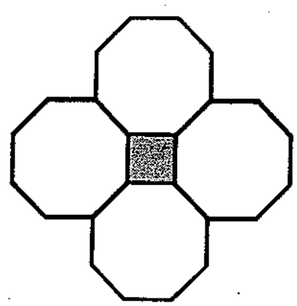
8. In the number spiral, give with proof a general formula for the numbers in the sequence, 1, 6, 19, ... that appear as part of row three as shown.

13	14	15	16	17			
12	3	4	5	18			
11	2	1	6	19	□	□	□
10	9	8	7	20			
		...	22	21			

9. Let R be the region consisting of points (x,y) in the plane satisfying both $|x|+|y| \le 2$ and $|y| \le 1$. Sketch the region R and find its area. (Reminder: $|x| = x$ if $x \ge 0$ and $|x| = -x$ if $x \le 0$.)

10. Find the minimum value of $$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$ for x>0. Give an explanation.

11. A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. The diagram shows the case $m = 4$, $n = 8$.

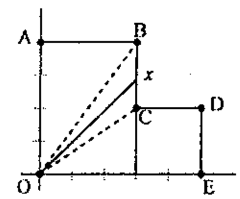


(a) If $m = 10$, what is the value of n?
(b) Determine with proof all possible (m,n).

SOLUTIONS – FINAL ROUND
January 31, 2005

1. a, b, c are 5, 9, 45; Trial and error does it. There is no point in trying 1 or 3 for a since $\frac{1}{1}$ and $\frac{1}{3}$ are too big. Try $a = 5$. Then $\frac{1}{b} + \frac{1}{c} = \frac{2}{15}$. Since $\frac{1}{7} > \frac{2}{15}$ there is no point in trying $b = 7$. Try $b = 9$ and you are done, with $c = 45$.

2. 8/9; The total area is 2x2+3x4=16; half is 8.



The area of Trapezoid OCDEO is 4+3=7; the area of the region OBCDEO is 4+6=10. The line 0x, where $x = (3, a)$ must bisect the area. Then $\frac{1}{2}(3a) + 4 = 8$ and $a = 8/3$. The slope of 0x is then 8/9.

Alternative solution: Subtract 4 from the 3x4 rectangle and bisect the resulting rectangular area which has base 3 and height $4 - \frac{4}{3} = \frac{8}{3}$.

3. (a) 8 (b) 2^{99} (c) 2^{n-1}; Consider $n = 1 + 1 + 1 + \ldots + 1$, a sum of n 1's, formed using n-1 plus signs. Selecting a plus sign can be viewed as forming a sum of the adjacent ones. For example, the sum 1 + 2 + 1 can be formed by selecting the second plus in 1 + 1 + 1 + 1 and not selecting the other two. Since there are two choices for each of the n-1 plus signs, there are 2^{n-1} ways of expressing n.

4. $n = 924$; $\frac{4}{11}n + \frac{k}{7}n + 324 = n$. Reasonable choices for k are 1, 2, 3. Try $k = 2$. Then $\left(\frac{4}{11} + \frac{2}{7}\right)n + 324 = n$ or $324 = n\left[1 - \frac{50}{77}\right] = n\frac{27}{77}$ and $n = 924$. More formally,

$28n + 11kn + 77(324) = 77n$, $(77)(324) = 49n - 11kn$, and $n = \frac{(77)(324)}{49 - 11k}$. The only possible k's are 1, 2, 3, 4 and of these only $k = 2$ produces an integer for n.

5. (a) $160,\ 80\sqrt{2}, 80, 40\sqrt{2}, 40, 20\sqrt{2}$

(b) The k-th perimeter, $P_k = 160\left(\frac{1}{\sqrt{2}}\right)^k$, is rational when k is even.

6. 25; First determine where the 1's are. The number of terms in each subsequence beginning with a 1 is 2, 4, 6, 8, 10… . For m = 1, 2, 3, …, 1's appear in positions $1+[2+4+6+8+\ldots+2(m-1)] = m^2 - m + 1$. When $m = 45$ there is a 1 in position 1981 [when m = 46, the next 1 holds position 2070 which is too big]. Count up until you hit the 2005th term.

7. (a) 30; List them. For each of the i choices of a, $1 \le i \le 10$, delete from the i choices for b the duplications; $(1+2+3+\ldots+10)-(1+3+5+7+9) = 10\cdot 11/2 - 5^2 = 30$.

(b) $\left[\frac{c+1}{2}\right]^2$ if c is odd, $\frac{c}{2}\left[\frac{c}{2}+1\right]$ if c is even; If c is odd,

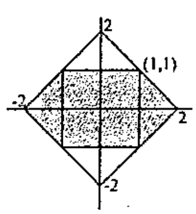
$$(1+2+\ldots+c)-2\left[1+2+\ldots+\frac{c-1}{2}\right]=\frac{c(c+1)}{2}-2\left[\frac{1}{2}\frac{c-1}{2}\left(\frac{c-1}{2}+1\right)\right]=\left(\frac{c+1}{2}\right)\left(c-\frac{c-1}{2}\right)=\left[\frac{c+1}{2}\right]^2.$$

If c is even $(1+2+\ldots+c)-(1+3+\ldots+(c-1))=\frac{c(c+1)}{2}-\left(\frac{c}{2}\right)^2=\frac{c}{2}\left[c+1-\frac{c}{2}\right]=\frac{c}{2}\left[\frac{c}{2}+1\right].$

Alternatively, when c is odd the total is $1+2+3+\ldots+q+q-1+q-2+\ldots 3+2+1=q^2$ where $q=\frac{c+1}{2}$.

8. $4n^2+n+1$; The sequence 1, 6, 19, 40, 69, 106, … has second differences a constant, so that a quadratic $a(n)=an^2+bn+c$ fits the data. If $a(0)=1$; $a(n)=4n^2+n+1$. If $a(1)=1$, $a(n)=4(n-1)^2+(n-1)+1=4n^2-7n+4$. A recursion for the sequence $a(0), a(1), \ldots$ is $a(n)=2a(n-1)-a(n-2)+8$ with $a(0)=1, a(1)=6$.

9. 6; The square has area 4, each triangle has area 1.



10. 6: Let $a=\left(x+\frac{1}{x}\right)^3$, $b=x^3+\frac{1}{x^3}$. Then $b^2=x^6+2+\frac{1}{x^6}$ and the original expression becomes

$$\frac{a^2-b^2}{a+b}=a-b=\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)=x^3+3x+\frac{3}{x}+\frac{1}{x^3}-x^3-\frac{1}{x^3}=3\left(x+\frac{1}{x}\right).$$ For

$x\geq 0$ a number plus its reciprocal is always at least 2. Proof: $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\geq 0$ implies

$x-2+\frac{1}{x}\geq 0$ or $x+\frac{1}{x}\geq 2$. Hence $3\left(x+\frac{1}{x}\right)\geq 6$. This last step can also be arrived at using the arithmetic-geometric means inequality: $\frac{a+b}{2}\geq\sqrt{ab}$ with $a=x, b=\frac{1}{x}$.

11. (a) $n=5$, (b) $(3,12),(4,8),(6,6)(10,5)$; Each interior angle of an m gon has measure $(180-360/m)$ and that of the n gon surrounding this m gon, $(180-360/n)$. Since two of the latter angles meet with one of the interior angles of the m gon to form 360° we have $(180-360/m)+2(180-360/n)=360.$. This becomes $(1-2/m)+2(1-2/n)=2$ or $mn-2n-4m=0$. Complete this as follows:

Method 1: Solve for n as $n=\frac{4m}{m-2}=\frac{4m-8+8}{m-2}=4+\frac{8}{m-2}$. Only $m=3, 4, 6, 10$ work. Method 2: $(m-2)(n-4)=8$ is the same as $mn-2n-4m=0$. Match up the factors of 8 as 1x8, 2x4, 4x2, 8x1.