UNIVERSITY OF NORTHERN COLORADO

MATHEMATICS CONTEST

First Round
For all Colorado Students Grades 7-12
November 5, 2005

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...
- The positive odd integers are 1, 3, 5, 7, 9, 11, ...
- The area of a triangle is 1/2 base times height.
- The area of a trapezoid is 1/2 height times the sum of the two bases.

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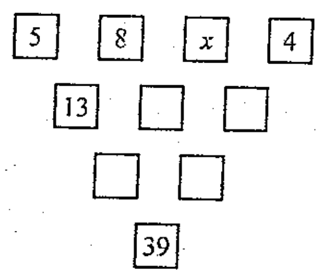
1. There are ten integers strictly between 7 and 18. Do not include 7 or 18.
   (a) How many integers are strictly between 48 and 65?
   (b) How many integers are strictly between the products ab and $(a+1)(b+1)$?

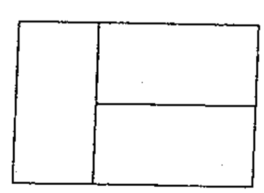
2. The number in an empty square box is found by adding the two numbers in the row directly above. As shown, $5+8=13$. What is the value of x?



3. The integer 12 has six positive divisors. They are 1, 2, 3, 4, 6 and 12.
   (a) Determine the number of positive divisors of 1008; include 1 and 1008.
   (b) Find the <u>smallest</u> multiple of 30 that has 36 divisors.

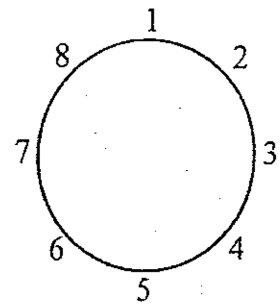
4. A unit fraction is a fraction of the form $\frac{1}{a}$, where a is a positive integer. Express $\frac{5}{11}$ as a sum of three different unit fractions with odd denominators. That is, find <u>different</u> odd integers a, b, and c so that $\frac{5}{11}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

5. Three congruent rectangles form a larger rectangle, as shown, with an area of 2400 cm^2. Determine the area of a square that has the same perimeter as the larger rectangle.



Over

6. The integers 1, 2, 3, 4, 5, 6, 7, 8 are written in order around a circle. Starting with 1, every second number (that is, 1, 3, 5, 7, 1, 3, ...) is underlined.



(a) Which integer is the 399^{th} to be underlined?

(b) If every third number is underlined (that is, 1, 4, 7, 2, ...), which is the 400^{th} number to be underlined?

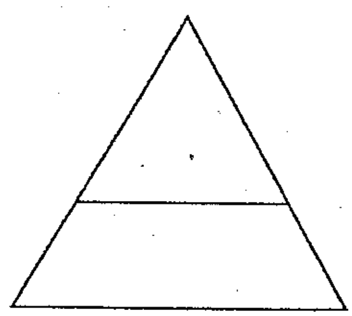
7. Let $T_n = \dfrac{n(n+1)}{2}$ be the n^{th} triangular number. For example, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$ and so on.

Express $\dfrac{1}{T_1} + \dfrac{1}{T_2} + \dfrac{1}{T_3} + \cdots + \dfrac{1}{T_{2005}}$ as a fraction $\dfrac{a}{b}$.

8. A jar contains a bunch of pennies, nickels, dimes and quarters. The average value of all these coins is 11¢. If you toss in one more nickel and three more pennies, the average value drops to 7¢. How many dimes were in the jar to begin with?

9. Four unequal positive integers a, b, c and N exist such that $N = 5a+3b+5c$. In addition $N = 4a+5b+4c$ and N is between 131 and 155. Determine the value of $a + b + c$.

10. A line intersects two sides of an equilateral triangle and is parallel to the 3^{rd} side. If the smaller triangle and the trapezoid formed have equal perimeters, determine the ratio of the area of the smaller triangle to that of the trapezoid. Express your answer as a fraction a/b.



11. How many triples of positive integers a, b, c, are there with $a < b < c$ and $a + b + c = 21$?

BRIEF SOLUTIONS TO FIRST ROUND
NOVEMBER 2005

1. (a) 16; $65-48-1=16$

(b) $a+b$; $(a+1)(b+1)-ab-1=a+b$. Note that in part (a) $a=4$, $b=12$ since $65=5\cdot13=(4+1)(12+1)$.

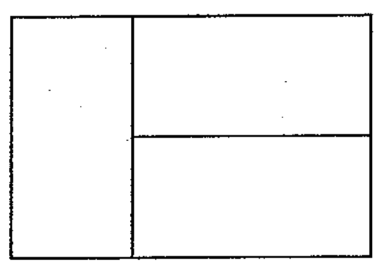
2. $x=2$; The entries in the four empty boxes are $x+8$, $x+4$, $x+21$ and $2x+12$. Then $(x+21)+(2x+12)=39$ or $3x+33=39$, and $x=2$.

3. (a) 30; $1008=2^4\cdot3^2\cdot7$ has $(4+1)(2+1)(1+1)=30$ divisors.

(b) 1260; Start with $30=2\cdot3\cdot5$, which has 8 divisors and build multiples to achieve 36 divisors. Since $36=2\cdot2\cdot3\cdot3$ it seems reasonable to look at expressions like $2^a\cdot3^b\cdot5^c\cdot7^d$. Trial and error produces $2^2\cdot3^2\cdot5\cdot7$ as the largest multiple of 30 with 36 divisors.

4. 3, 9, and 99 or 3, 11 and 33; First subtract the largest unit fraction possible with odd dominator $\frac{5}{11}-\frac{1}{3}=\frac{4}{33}$. Repeat with $\frac{4}{33}$ yielding $\frac{4}{33}-\frac{1}{9}=\frac{1}{99}$. Then $\frac{5}{11}=\frac{1}{3}+\frac{1}{9}+\frac{1}{99}$. If you first subtract $\frac{1}{3}$ and then $\frac{1}{11}$ you get the second solution.

5. 2500cm^2; Since $x(x+y)=2400$ and $x=2y$, $6y^2=2400$ gives $y=20$ and $x=40$. The perimeter is then $4x+2y=160+40=200$ and the desired square has side 50; area $=2500\text{cm}^2$.



6. (a) 5; The sequence repeats the cycle 1, 3, 5, 7. After 99 cycles the third number, 5, is underlined.

(b) 6; The sequence repeats the cycle 1, 4, 7, 2, 5, 8, 3, 6. So 6 appears in positions numbered 8, 16, 24, …, 400.

7. $\frac{2005}{1003}$; $\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}+\cdots+\frac{1}{T_{2005}}=\frac{2}{1\cdot 2}+\frac{2}{2\cdot 3}+\cdots+\frac{2}{2005\cdot 2006}=$

$$2\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2005}-\frac{1}{2006}\right)\right]=2\left[1-\frac{1}{2006}\right]=\frac{2005}{1003}.$$

This telescoping process used the partial fraction decomposition $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$.

8. 2; If the jar contained n coins the total value would be $.11n$. After tossing in one nickel and three pennies the total value would be $.11n+.08$ which must equal $.07(n+4)$ resulting in $n=5$ coins. You can only make $5(.11)=55$ cents using 2 nickels, 2 dimes and one quarter.

9. $a+b+c=33$; Subtract $4\text{N}=20a+12b+20c$ from $5\text{N}=20a+25b+20c$ and $\text{N}=13b$. The only multiple of 13 between 131 and 155 is $\text{N}=143=13\cdot 11$; so $b=11$. From $143=5a+33+5c$ we get $a+c=22$ and $a+b+c=33$.

10. $\frac{9}{7}$; Let s be the length of a side of the smaller triangle, and $s+t$ the length of a side of the larger triangle. The area of the smaller triangle is $\frac{\sqrt{3}}{4}s^2$. The area of the larger triangle is $\frac{\sqrt{3}}{4}(s+t)^2$. Then the desired ratio is $\left[\left(\sqrt{3}/4\right)s^2\right]\div\left(\left(\sqrt{3}/4\right)\left[(s+t)^2-s^2\right]\right)=s^2/(2st+t^2)$. Now use $3s=s+2t+s+t$ or $s=3t$ to obtain $9/7$.

11. 27; Letting a vary from 1 to 6 and counting you get $8+7+5+4+2+1=27$.

MATHEMATICS CONTEST

FINAL ROUND

For Colorado Students Grades 7-12

January 28, 2006

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...
- (Geometric) probability equals favorable area divided by total area.
- The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, ...
- The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

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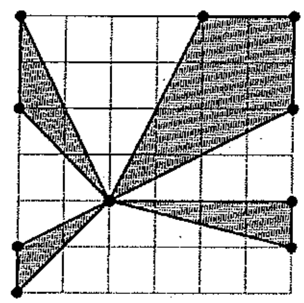
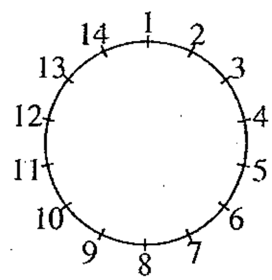
1. If a dart is thrown at the six by six target (and hits it) what is the probability that it will hit the shaded area?



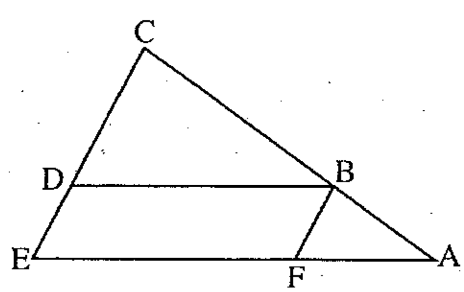
2. If a, b, and c are positive integers, how many integers are strictly between the products abc and $(a+1)(b+1)(c+1)$? For example, there are 35 integers strictly between 24 and 60.

3. The first 14 integers are written in order around a circle. Starting with 1 every fifth integer is underlined. (That is, 1, 6, 11, 2, 7, ...). What is the 2006[th] number underlined?



4. Determine all positive integers n such that n^2+3 divides evenly (without remainder) into n^4-3n^2+10.

5. In the figure BD is parallel to AE, and also BF is parallel to DE. The area of the larger triangle ACE is 128. The area of the trapezoid BDEA is 78. Determine the area of the triangle ABF.



Over

6. The sum of all the positive integer divisors of $6^2 = 36$ is $1+2+3+4+6+9+12+18+36=91$.
   (a) Determine a nice closed (ie, without dots or the summation symbol) formula for the sum of all the positive integer divisors of 6^n.
   (b) Repeat for 12^n.
   (c) Generalize.

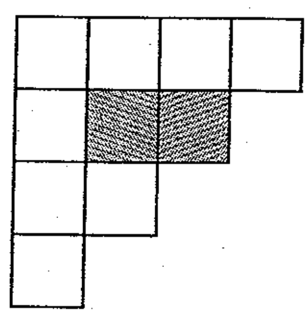
7. The five digits *a*, *b*, *c*, *d*, and *e* of 55225 are such that $a = b = e$ and $c = d$; in addition, $55225 = 235^2 = (235)(235)$. Find another positive integer m such that m^2 is also a five-digit number $abcde$ that satisfies $a = b = e$ and $c = d$.

8. Find all positive integers n so that $n^3 - 12n^2 + 40n - 29$ is a prime number. For each of your values of n compute the value of this cubic polynomial showing that it is, in fact, a prime.

9. Determine three positive integers *a*, *b*, and *c* that simultaneously satisfy the following three conditions:
   (i) $a < b < c$,
   (ii) Each of $a+b$, $a+c$, $b+c$ is the square of an integer, and
   (iii) c is as small as possible.

10. How many triples of positive integers *a*, *b*, and *c* are there with $a < b < c$ and $a+b+c = 401$?

11. Call the figure to the right a 4-tableau shape. Determine the number of rectangles of all sizes contained in this shape. Note that a square is a rectangle and that a 1x2 rectangle (shaded) is different from a 2x1 rectangle. Express your answer as a binomial coefficient, and explain the significance of your expression. Generalize, with proof, to an *n*-tableau shape.



1. $\frac{13}{36}$; The area of the shaded region is 1 + 2 + 8+ 2 =13.

2. $ab+ac+bc+a+b+c$; compute $(a+1)(b+1)(c+1)-abc-1$

3. 2 ; The sequence 1, 6, 11, 2, 7, 12, 3, 8, 13, 4, 9, 14, 5, 10, 1, 6, ... repeats in blocks of 14. Since $2006 = 4+143\cdot 14$, 2 is the 2006th term underlined. Numbers of the form $14k$ land on 10.

4. 1, 2, 5 ; Using long division of polynomials $n^4-3n^2+10=(n^2-6)(n^2+3)+28$ or $(n^4-3n^2+10)/(n^2+3)=(n^2-6)+28/(n^2+3)$. Now try possible values for n. Only $n=1$, $n=2$ and $n=5$ provide factors of 28.

5. 18; In general, if G and H are similar triangles with two corresponding sides s_1 and s_2 then Area G/Area H = s_1^2/s_2^2 . Here BCD is $50/128 = 25/64$ of ACE. So BC is $5/8$ of AC or AB is $3/8$ of AC. Then ABF is $9/64$ of ACE or $(9/64)128=18$.

6. (a) $(2^{n+1}-1)(3^{n+1}-1)/2$; Each divisor of 6^n is a term in the expansion of $(1+2+\cdots+2^n)(1+3+3^2+\cdots+3^n)$. Now sum these geometric sums.

   (b) $(2^{2n+1}-1)(3^{n+1}-1)/2$; Each divisor of $12^n=2^{2n}\cdot 3^n$ is a term in $(1+2+2^2+\cdots+2^{2n})(1+3+3^2+\cdots+3^n)$.

7. 109; m must be between 100 and 320 to be a 5-digit number. Since $a=e$, the earliest possible values of m are 101, 109, 111, 119, ... since only $1^2=1$ and $9^2=81$ end in a 1. The second one you try works, namely $m=109$.

8. 2, 4 ; Since $n^3-12n^2+40n-29=(n-1)(n^2-11n+29)$ either $n-1=1$ or $n^2-11n+29=1$ and the other factor is prime. If $n-1=1$, $n=2$ and $2^2-11\cdot 2+29=11$, a prime. If $n^2-11n+29=1$, $n^2-11n+28=0$ or $(n-4)(n-7)=0$. Testing each of $n=4$ and 7, only $n=4$ makes $n-1$ a prime.

9. $a=6$, $b=19$, $c=30$; Starting with $1-3-6$, (which only partially works), it's easy to see that $a=1$ fails. Same with $a=2$. There are more options with $a=3$, but these also fail. $a=4$ and $a=5$ are more profitable and $5-20-44$ will work. But $c=44$ is not minimal. Continuing with $a=6$ we find that $6-19-30$ works.

But a=2 works 2-34-47

10. 13,200 ; Counting all triples (a, b, c) from (1, 2, 398) to (132, 134, 135) creates an "almost" arithmetic sum, one with "holes" in every 3rd position: $2+3+5+6+8+9\cdots+195+197+198$. The "holes" are at 4, 7, 10, ..., 196. Take $(2+3+4+\cdots+198)-(4+7+10+\cdots+196)=[200(197)-200(65)]/2=200(132)/2=13{,}200$.

11. $\binom{7}{4}$, $\binom{n+3}{4}$ in general. The combinatorial proof of this and related questions is being submitted in an article to the MT.