NORTHERN COLORADO MATHEMATICS CONTEST

November 14, 1992

The ten digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, ...

You have 3 hours. Work as many of the problems as you can on your own scratch paper, and then transfer your answers to the ANSWER SHEET.

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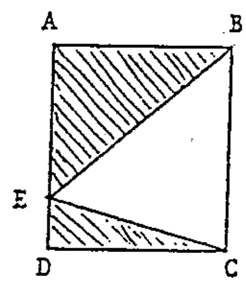
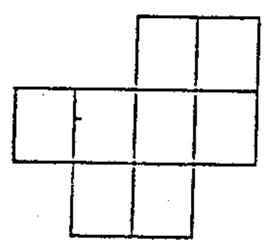
1. ABCD is a square with side length 12 (that is, AB = 12). What is the area of the shaded region?



2. The repeating decimal expansion for 1/99 is .01010101..., with 1's in the even numbered decimal places and 0's in the other places. What is the digit in the 16th decimal place of the decimal expansion for

$$\frac{1}{9}+\frac{1}{99}+\frac{1}{999}?$$

3. The figure consists of eight identical squares having <u>total</u> area of 392 in^2. Determine the perimeter of the figure.



4. The table gives some of the straight line distances (that is, as the crow flies) between certain pairs of cities. For example, the distance between city A and city C is 17. Use the given data to find the distance between A and B.

	A	B	C	D
A			17	
B				14
C		8		
D	23		6	

5. The notation n! (read n factorial) represents the product of all whole numbers from 1 to n, that is, n!=1·2·3·4...n. For example, 4!=1·2·3·4=24 and 7!=1·2·3·4·5·6·7. Find three

$$4 \cdot 5 \cdot 6 = \frac{a!}{b!}$$

6. Starting with 8, arrange the fifteen integers 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15 in a row so that the sum of any two adjacent integers is a perfect square.

7. In solving a quadratic equation, Cardano misread the coefficient of x and obtained 3 and -4 as solutions, while Ferrari misread the constant term and obtained -1 and 5 as solutions. Find the correct solutions.

8. In a magic square, the sum of every row, column, and diagonal is the same. This sum is called the <u>magic sum.</u> In FIGURE 1, this magic sum is 15. Determine the value of A + D in the magic square given in FIGURE 2.

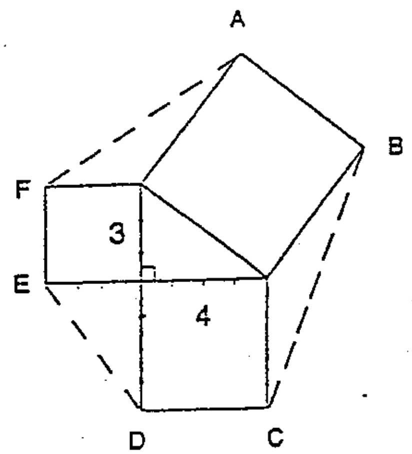
4	9	2
3	5	7
8	1	6

FIG. 1

19	A	14
10	B	C
D	E	F

FIG. 2

9. The figure shows a right triangle having sides of length 3 and 4 with three squares on its sides. The vertices of the squares are joined to form three triangles. Determine the area of the hexagon ABCDEF.



10. The integer 20 can be written as the sum of positive integers in many ways. Here are a few possible ways: 20 = 10 + 10, 20 = 5 + 7 + 8, 20 = 1 + 2 + 5 + 12. Forming the products of the summands in these three cases yields 10·10 = 100, 5·7·8=280, and 1·2·5·12=120. The largest product here is 280.

(a) How should you express 20 as a sum of positive integers so that the product of the summands is as large as possible?

(b) What is that largest product?

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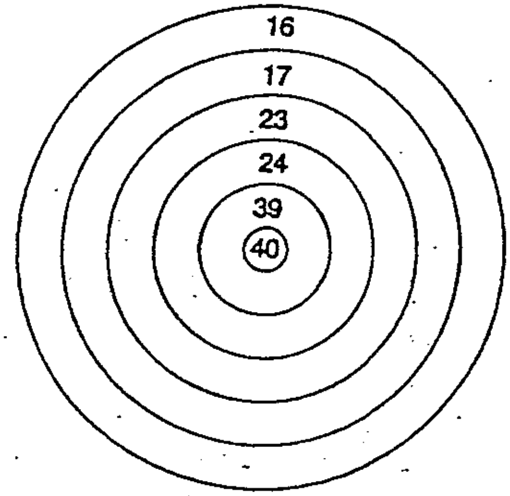
1. How many times must one shoot at this target, and which rings must one hit in order to score exactly 100 points?



2. Determine the digit in the 623rd place after the decimal point in the repeating decimal for:

$$\frac{1}{9}+\frac{2}{99}+\frac{3}{999}.$$

3. A student thinks of four numbers. She adds them in pairs to get the six sums 9, 18, 21, 23, 26, 35. What are the four numbers? There are two different solutions.

4. The table gives some of the straight line distances between certain pairs of cities. For example, the distance between city A and city B is 34. Use the given data to determine the distance between city A and city C. (Hint: a problem on the First Round was similar in spirit to this one.)

	A	B	C	D
A		34		16
B			42	
C				12
D		30		

5. A collection of 25 consecutive positive integers adds to 1000. What are the smallest and largest integers in this collection.

6. Observe that

$$3^2 + 4^2 + 12^2 = 13^2$$

$$4^2 + 5^2 + 20^2 = 21^2$$

(a) Find integers x and y so that $5^2 + 6^2 + x^2 = y^2$.

(b) Conjecture a general rule that is being illustrated here.

(c) Prove your conjecture.

7. Choose four numbers by circling exactly one number in each horizontal row and exactly one number in each vertical column. Compute the product of these four numbers. Explain clearly why the same product results no matter which selection of this type of four numbers you make.

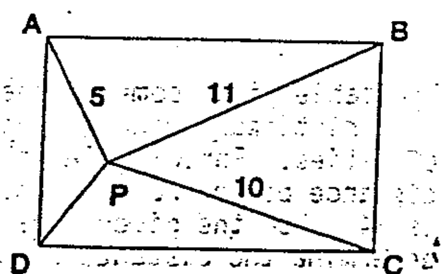
10	15	35	20
2	3	7	4
8	12	28	16
12	18	42	24

8. For what integer value of n is the expression

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots + \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

equal to 7? (Hint: $(\sqrt{1}+\sqrt{2})(\sqrt{1}-\sqrt{2}) = -1$)

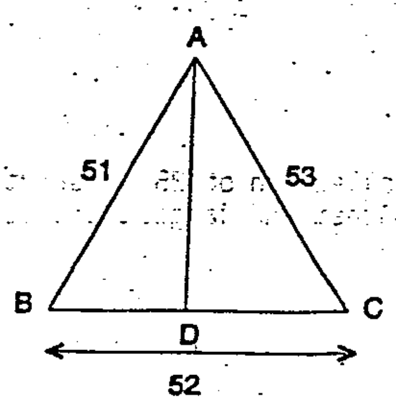
9. Let P be a point inside the rectangle ABCD. If AP = 5 BP = 11 and CP = 10, find the length DP. (Hint: Draw helpful horizontal and vertical lines.)



10. The scalene triangle ABC has side lengths 51, 52, 53. AD is perpendicular to BC.

(a) Determine the length of BD.

(b) Determine the area of triangle ABC.



1. Shoot 6 times hitting 2 16's and 4 17's; $2 \cdot 16 + 4 \cdot 17 = 100$.

2. $1/9 = .1111111...$
$2/99 = .0202020...$
$3/999 = .0030030...$

The sum in decimal form is .134316134316...

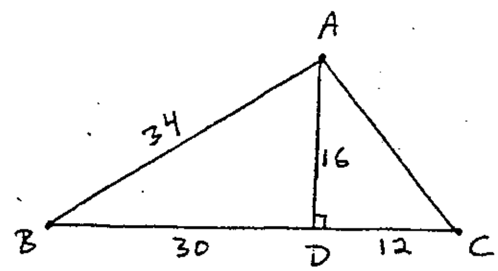
The largest multiple of 6 less than 623 is 618. Hence the digit in the 623rd place is the 5th digit in the block 134316, namely, 1.

3. The two solutions are 2, 7, 16, 19 and 3, 6, 15, 20. If we let a, b, c, d denote the four numbers we could write:

$a + b = 9$		$a + b = 9$
$a + c = 18$		$a + c = 18$
$a + d = 21$	or	$a + d = 23$
$b + c = 23$		$b + c = 21$
$b + d = 26$		$b + d = 26$
$c + d = 35$		$c + d = 35$

In each case subtract the second line from the first obtaining $b - c = -9$. Now combine with $b + c = 23$ or $b + c = 21$, yielding $b = 7$ or $b = 6$ and the two solutions follow.

4. B, C, D lie on a line. The four cities appear as in the figure. $\angle ADB$ is a right angle since $34^2 = 30^2 + 16^2$. Then $AC = 20$.



5. Here are two solutions:

(a) The middle number of the 25 numbers must be 40. Hence, the smallest number is $40 - 12 = 28$; the largest is $40 + 12 = 52$.

(b) $n + (n+1) + (n+2) + ... + (n+24) = 1000$ gives $25n + (1+2+...+24) = 25n + 300 = 1000$ and $n = 28$. The smallest is 28, the largest is 52. Recall that the closed formula for the sum $1+2+3+...+n$ can be found, as Gauss did, by writing it backwards and adding. Try this!

6. (a) $x = 30$, $y = 31$

(b) $n^2 + (n+1)^2 + [n(n+1)]^2 = [n(n+1) + 1]^2$

(c) $[n(n+1)+1]^2 = [n(n+1)]^2 + 2n(n+1) + 1 = [n(n+1)]^2 + 2n^2 + 2n + 1 = [n(n+1)]^2 + n^2 + (n+1)^2$

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