NORTHERN COLORADO MATHEMATICS CONTEST

For grades 7 – 12

FIRST ROUND

The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 ,14, 15, 16, 17, . . .

You have 3 hours. Work as many problems as you can on your own scratch paper, and then transfer your answers to the ANSWER SHEET.

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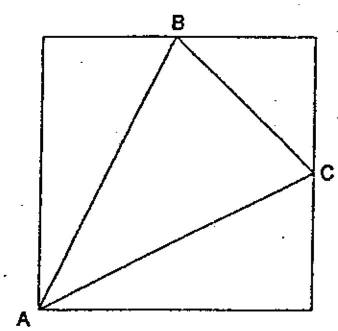
1. What is the 100th term of the arithmetic progression $9, 16, 23, 30, 37, 44, \ldots$ in which 7 is the difference between successive terms?

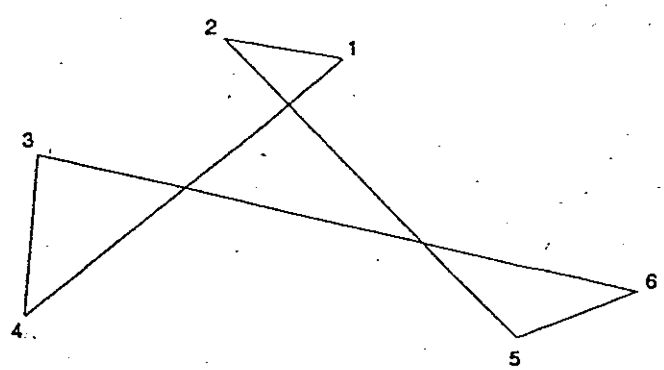
2. The integer 6 has four positive integer divisors (including 1 and 6), namely $1, 2, 3$ and 6. Determine the number of positive integer divisors for each of the following numbers:
(a) $40 = 2^3 \cdot 5$ (b) 720

3. The sum of two numbers is 6 and their product is 2. What is the sum of their reciprocals? (The reciprocal of x is $\frac{1}{x}$.)

4. The square at the right has sides 12 inches long. Points B and $C$are the midpoints of their respective sides. What is the area of ΔABC?



5. Determine the sum of the six angles labeled in the figure.



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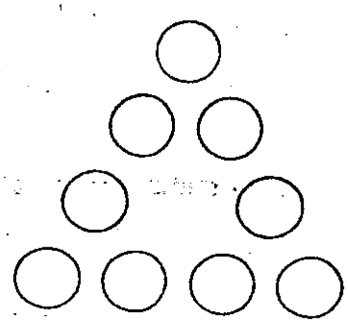
6. On a tile floor with square tiles of side 1 foot, a circle of radius $r = 3.2$ feet is drawn with center at one corner of a tile. How many corners are included inside the circle?

7. Place the integers $1, 2, ..., 9$ in the nine circles so that the four integers along each of the three sides add up to the same total. What is the smallest value this total can have?



8. The three sides of a triangle have lengths a, b and c. Each is an integer and $a \leq b \leq c$. If $c = 7$ how many such triangles are possible? (Hint: $2, 3$ and 7 cannot be lengths of sides of a triangle.)

9. The average of five integers is their sum divided by 5. The median of five integers is the middle integer after the integers have been arranged in order. For example, the average of $3, 4, 1, 11$, and 6, is 5 and the median is 4 (since the rearrangement of these integers is $1, 3, 4, 6, 11$ and the middle integer is 4).
Determine all possible values of the integer m such that the median and average of the five integers $3, 6, 9, 11$, and m are the same.

10. Determine the positive integer n such that

$$\frac{1+3+5+7+...+(2n-1)}{2+4+6+8+10+...+2n} = \frac{115}{116}$$

11. A line segment joining the points $(0, 120)$ and $(200, 0)$ contains how many points having both coordinates as integers? In your count, include the two endpoints $(0, 120)$ and $(200, 0)$.

NORTHERN COLORADO MATHEMATICS CONTEST

FINAL ROUND March 5, 1994

The positive integers are the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, . . .

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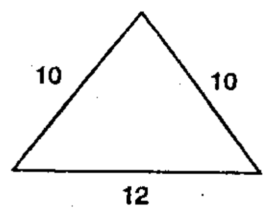
1. Find three different positive integers given that their product is 72 and that their sum is one of the numbers in the set $\{7, 14, 21, 28, 35\}$. [Hint: Consider all factorizations of 72 into three factors, including factorizations such as $72 = 1 \times 2 \times 36$.]

2. An unending arithmetic progression is generated by adding a fixed number to a term to obtain the next term. Find the smallest number that is a term in each of the two following unending arithmetic progressions:

   (a) 4, 11, 18, 25, . . .

   (b) 5, 13, 21, 29,. . .

3. Find the area of the isosceles triangle whose sides have lengths $10, 10, 12$. [Hint: Cut the triangle into two right triangles.]



4. If the indicated triangular arrangement of the positive integers is continued,
   (a)in which row will the integer 45 appear?
   (b)in which row will 999 appear?

1	Row 1
2 3	Row 2
4 5 6	Row 3
7 8 9 10	Row 4
11 12 13 14 15	Row 5
⋮	⋮

5. The partial table

n	1	2	3	4	5	6	...
positive integer divisors of n	1	1, 2	1, 3	1, 2, 4	1, 5	1, 2, 3, 6	...
number of such divisors	1	2	2	3	2	4	...

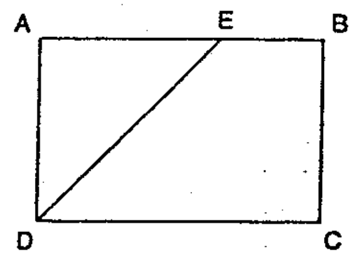
shows that 6 is the smallest positive integer with exactly 4 positive integral divisors.

(a) What is the smallest positive integer with exactly 6 positive integral divisors?

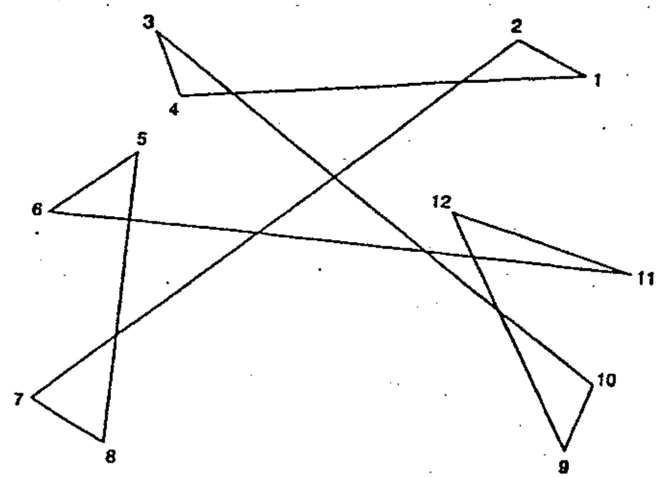
(b) What is the smallest positive integer with exactly 10 positive integral divisors?

6. Given that $x + y = 1$ and $x^2 + y^2 = 2$, find $x^4 + y^4$. [Warning: The answer is not 4.]

7. In the given figure, $ABCD$ is a rectangle, with AD=5. Point E is on side AB such that $EB = 3$ and $DE = DC$. Find DC.



8. Determine the sum of the measures (in degrees or radians) of the twelve angles labeled with the numbers $1, 2, 3, ..., 12$.



9. For positive integers c, we use the notation N_c to stand for the number of pairs $\{a, b\}$ of positive integers such that simultaneously a is less than b, b is less than c, and c is less than $a + b$.

The partial table

c	4	5	6	...
pairs	$\{2,3\}$	$\{2,4\}, \{3,4\}$	$\{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}$	...

shows that $N_4 = 1$, $N_5 = 2$, $N_6 = 4$. Find:

(a)N_8 (b)N_{10} (c)N_{12} (d)N_{98} (e)N_{99}

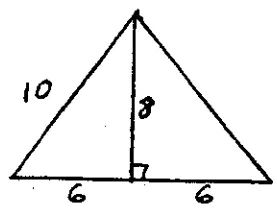
PROBLEM 10 WILL BE USED ONLY TO BREAK TIES

10. As clearly as you can, justify your answer to problem 9(e).

1. The integers are 1, 3 and 24. These can be determined through trial and error noting that their sum is a multiple of 7.

2. The smallest number common is 53. The common differences are 7 and 8 respectively.

3. Area= 48. Using the Pythagorean Theorem, we can determine that the altitude is 8. Then $\frac{1}{2}(\text{altitude}) \times (\text{base}) = \frac{1}{2}(8)(12) = 48$.



4. (a) 45 appears on the 9^{th} row. The rows end with the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45,. . . and 45 is the 9^{th} number listed.

(b)999 appears on row 45. From the table of data,

row #	1	2	3	4	5	6	7	8	9
# at end of row	1	3	6	10	15	21	28	36	45

and the observations that

$$6 = \tfrac{1}{2}(3 \cdot 4)$$
$$10 = \tfrac{1}{2}(4 \cdot 5)$$
$$15 = \tfrac{1}{2}(5 \cdot 6)$$
$$21 = \tfrac{1}{2}(6 \cdot 7)$$

We see that row n ends with the number $\frac{n(n+1)}{2}$. Trial and error shows that when $n = 44$, $\frac{n(n+1)}{2}$ has value $\frac{44(45)}{2} = 990$. Then 999 must be on row 45.

5. (a) 12; each divisor of 12 is of the form 2^a3^b since 2 and 3 are the only prime divisors of 12. The power a can be 0, 1 or 2, and the power b can be 0 or 1. Since there are 3 choices for a and 2 choices for b, 12 has 6 divisors and 12 is the smallest.

(b)48. The desired integer looks like $2^a \cdot 3^b \cdot 5^c$... since $10 = 5 \cdot 2$, if $a = 4$ and $b = 1$, $2^4 \cdot 3^1 = 48$ would be the smallest integer with exactly 10 divisors.

6. $3\frac{1}{2}$. $4 = (x^2+y^2)^2 = x^4+y^4+2x^2y^2 = x^4+y^4+2[\frac{1}{4}]$ since $1 = (x+y)^2 = x^2+y^2+2xy = 2+2xy$, yielding $xy = -\frac{1}{2}$, and hence, $x^2y^2 = \frac{1}{4}$. Then $x^4+y^4 = 4 - \frac{1}{2} = 3\frac{1}{2}$.