THE UNIVERSITY OF NORTHERN COLORADO MATHEMATICS CONTEST

FOR GRADES 7 - 12

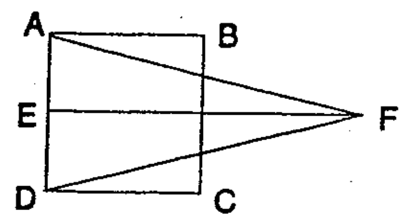
NOVEMBER 5, 1994

The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22 ... Work as many of the problems as you can on your own scratch paper, and then transfer your answers to the ANSWER SHEET. Calculators are permitted but are not necessary. You have three hours.

****

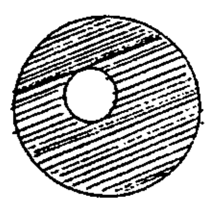
1. A textbook is opened at random. What are the page numbers if the product of the two facing pages is 2256?

2. The isosceles triangle AFD and the square ABCD have the same area. If AB=5 cm determine the length of the altitude FE.

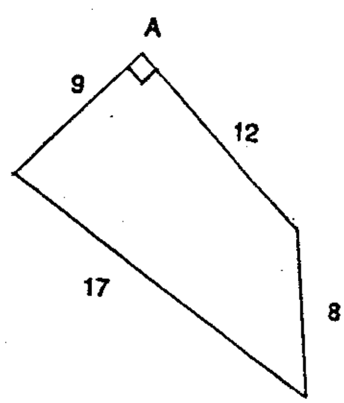


3. The average of seven different positive integers is 7. What is the largest possible value of any of these numbers?

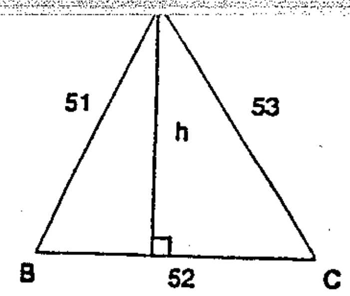
4. The area of the shaded region is 72π cm^2. If the radius of the small circle is 3 cm, what is the radius of the larger circle?



5. Determine the area of the quadrilateral with given lengths and a right angle at A. All measurements are in meters.



6. What is the altitude h of triangle ABC? Express your answer as an integer.

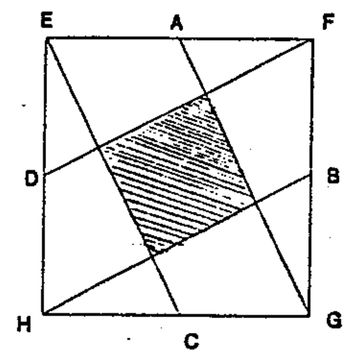


7. Express the polynomial $n^4 + 2n^3 + 2n^2 + 2n + 1$ as a product of two quadratics.

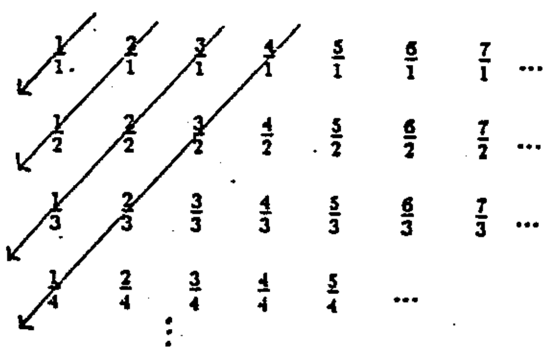
8. (a) One factor of $x^3 + 1$ is $(x + 1)$. What is the other factor?

(b) Show that $8^{17} + 1$ is composite by expressing $8^{17} + 1$ as a product of two factors. (Hint: You should leave the terms in your factors in exponential form.)

9. In the figure, A, B, C, and D are midpoints of sides of the square EFGH. If EF=8 cm, what is the area of the smaller shaded inner square?



10. Each positive rational number can be found in the scheme to the right: Reading diagonally down the arrows, the following natural correspondence with the positive integers emerges:



$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{3}{1}$	$\frac{2}{2}$	$\frac{1}{3}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1	2	3	4	5	6	7	8	9	10	...

Ignoring the fact that some of the rational numbers are not in reduced form we can say, for example, that 2/3 is the 9th rational number.

(a) Which positive integer corresponds to the fraction 7/12?

(b) Which fraction corresponds to the integer 1415?

11. Determine the two different values of $x^3 + \frac{1}{x^3}$ given that $x^2 + \frac{1}{x^2} = 14$. (Hint: examine $(x + \frac{1}{x})^3$)

UNC MATHEMATICS CONTEST
FIRST ROUND SOLUTIONS

NOVEMBER 1994

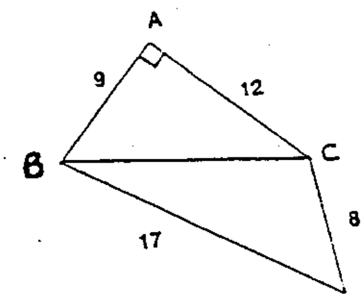
1. Since the page numbers are consecutive integers, they must end in 2 and 3 or 7 and 8. Trial shows $47 \times 48 = 2256$. Or, solve $x(x + 1) = 2256$.

2. The area of ABCD is 25. Then $\frac{1}{2}(5)(FE) = 25$ implies $FE = 10$ cm.

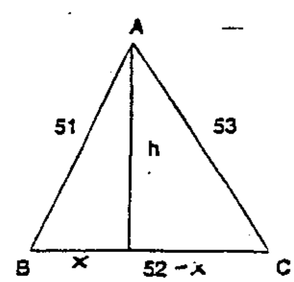
3. From $\frac{1+2+3+4+5+6+f}{7} = 7$ we see that $21 + f = 49$ and $f = 28$.

4. The area of the large circle is $72\pi + 9\pi = 81\pi$ cm^2. Then it's radius is 9 cm.

5. Draw line BC. Since $\triangle ABC$ is right the Pythagorean theorem says that $BC = 15$. Since $17^2 = 8^2 + 15^2$, $\triangle BCD$ is right. Finally the area of the quadrilateral ABDC is $\frac{1}{2}(9)(12) + \frac{1}{2}(8)(15) = 114$ m^2.



6. Subtracting $51^2 = h^2 + x^2$ from $53^2 = h^2 + (52-x)^2$ gives: $53^2 - 51^2 = (52-x)^2 - x^2$ and $(53-51)(53+51) = 52^2 - 104x$ or $2(104) = 52^2 - 104x$ or $4 = 52 - 2x$, and $x = 24$. Finally from $51^2 = h^2 + 24^2$, $h = 45$. Alternatively one could use Heron's Formula.



7. $n^4 + 2n^3 + 2n^2 + 2n + 1 = (n^4 + 2n^2 + 1) + (2n^3 + 2n) = (n^2 + 1)^2 + 2n(n^2 + 1)$ $= (n^2 + 1)[n^2 + 1 + 2n] = (n^2 + 1)(n + 1)^2$. Or notice that -1 is a root of $n^4 + 2n^3 + 2n^2 + 2n + 1 = 0$ and divide out $(n + 1)$ by synthetic division.

MATHEMATICS CONTEST

FINAL ROUND   Feb 4, 1995   1-4 PM

The positive integers are the numbers 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,...

The area of a circle of radius r is: πr^2.

A formula that appears frequently in mathematics is: $1+2+3+...+n = n(n+1)/2$.

---

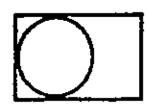
1. List all the integers from 1 to 82 that have an odd number of divisors. For example, 9 has three divisors: 1,3 and 9, but 10 has an even number of divisors: 1,2,5 and 10.

2. Each student in a group of more than 2 but fewer than 20 students paid the same whole dollar amount for a concert ticket. If the total cost for the group was \$551, how much was each ticket?

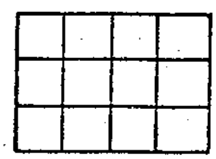
3. a) List all the postage amounts that cannot be made using just 3¢ and 7¢ stamps. For example, 17¢ worth of postage can be made using two 7¢ stamps and one 3¢ stamp, but the value 5¢ cannot be made.

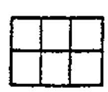
   b) Explain why all other values can be made.

4. A circle is tangent to three sides of a rectangle. If the area of the rectangle is twice the area of the circle, what is the ratio of the longest side of the rectangle to the shortest side?



5. a) How many rectangles of all sizes (include the 1 by 1, 2 by 2, etc.) are there in the 3 by 4 rectangle to the right? Here, a 1 by 2 rectangle is considered to be different than a 2 by 1 rectangle, for example. Before beginning this problem, you might check that the 2 by 3 rectangle contains 18 rectangles of all sizes.





   b) How many rectangles of all sizes are there in an **m by n** rectangle where m and n are integers?

6. A piece of wire that is 104 cm long is cut into two pieces and two squares are formed. The sum of the areas of the two squares is 388 cm^2; the sides have integral lengths. What are the dimensions of each square?

(over)

7. Factor $n^4 - 20n^2 + 4$ as a product of two quadratics with integer coefficients.

8. Doors to hotel rooms along a long straight corridor are numbered 1 through 500. 500 guests engage in the following exercise:

   Guest #1 opens all the doors
   Guest #2 closes every other door, starting with door #2; that is, she closes doors numbered 2,4,6,8,...,498,500.
   Guest #3 changes the position of every third door, starting with door #3 (that is, she opens it if closed, and closes it if open).
   Guest #4 changes the position of doors numbered 4,8,12,16...
   AND SO ON...

   a) After guest **#500** completes his task, which doors are open?

   b) Explain your reasoning.

9. In the pattern to the right we see that cubes can be written as sums of consecutive odd integers.

   $1^3 = 1$

   $2^3 = 3 + 5$

   $3^3 = 7 + 9 + 11$

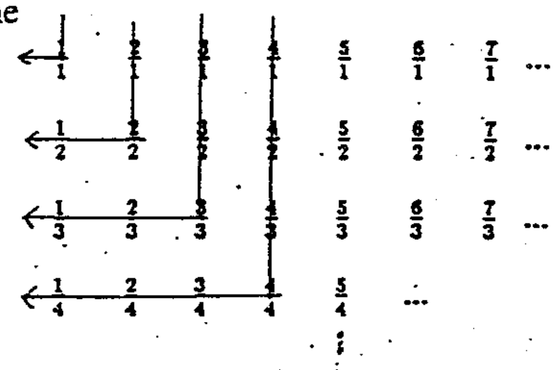
   $4^3 = 13 + 15 + 17 + 19$

   a) Express 5^3 as a sum of five consecutive odd integers.

   b) Give a formula for expressing n^3 as a sum of consecutive odd integers.

   c) Prove your rule.

10. Each positive rational number can be found in the scheme to the right. Reading along the "angled" arrows, the following natural correspondence emerges:



$\frac{1}{1}$	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{1}{2}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	...
↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	2	3	4	5	6	7	8	9	...

Ignoring the fact that some of the rational numbers are not in reduced form, we can say, for example, that 1/3 is the 9th rational number.

a) Which positive integer corresponds to the fraction 5/7?

b) Which fraction corresponds to the integer 792?

c) Which integer corresponds to the fraction m/n? Explain your reasoning.

SOLUTIONS FINAL ROUND

FEBRUARY 4, 1995

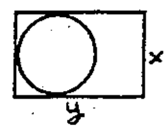
1. 1, 4, 9, 16, 25, 36, 49, 64, 81. Only perfect squares have an odd number of divisors.

2. Each ticket costs \$29 since $551 = 19 \times 29$.

3. (a) $\{1, 2, 4, 5, 8, 11\}$.

(b) Since 12, 13, and 14 cents can be made, just add a 3 cent stamp to each, creating 15, 16, 17 and then 18, 19, 20.

4. $\frac{\pi}{2}$; $xy = 2\pi(\frac{x}{2})^2$ implies $\frac{y}{x} = \frac{\pi}{2}$.



5. (a) The following chart shows the count for each size:

1×1	12	2×1	8	3×1	4
1×2	9	2×2	6	3×2	3
1×3	6	2×3	4	3×3	2
1×4	3	2×4	2	3×4	1

$$\begin{aligned}\text{Total} &= (1+2+3+4)+(2+4+6+8)+(3+6+9+12)\\ &= (1+2+3+4)+2(1+2+3+4)+3(1+2+3+4)\\ &= (1+2+3)(1+2+3+4) = 60\end{aligned}$$

(b)

1×1	mn	2×1	$(m-1)n$	$\cdots$
1×2	$m(n-1)$	2×2	$(m-1)(n-1)$	$\cdots$
1×3	$m(n-2)$	2×3	$(m-1)(n-2)$	$\cdots$
$\cdots$		$\cdots$		
$1 \times n$	m	$2 \times n$	$(m-1)$	$\cdots$

$= m(1 + 2 + ... + n) + (m - 1)(1 + 2 + ... + n) + ... + 1(1 + 2 + ... + n)$

$= (1 + 2 + ... + n)(1 + 2 + ... + m) = \frac{n(n+1)}{2}\frac{m(m+1)}{2}$, the product of two triangular numbers.