UNIVERSITY OF NORTHERN COLORADO STATEWIDE MATHEMATICS CONTEST

FIRST ROUND NOVEMBER 1995

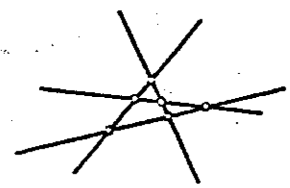
The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,...
Some useful formulas include:

$1 + 2 + 3 + \ldots + n = n(n+1)/2$

The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b, c are the appropriate sides of a right triangle.

****

1. The greatest number of points of intersection possible for four lines in a plane is 6.

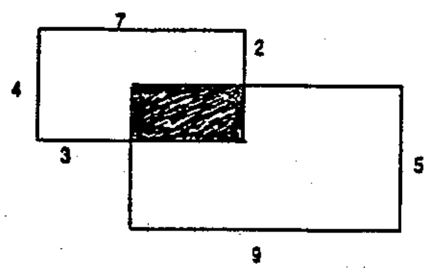


a) Fill in the following chart showing the greatest number of points of intersection for two, three, five, and six lines.

# of lines	2	3	4	5	6
# of points of intersection			6		

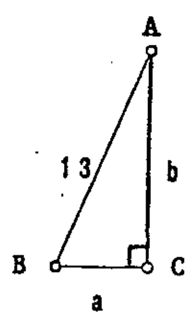
b) What is the greatest number of points of intersection for twenty lines?

2. What is the total area of the unshaded parts of the two rectangles?

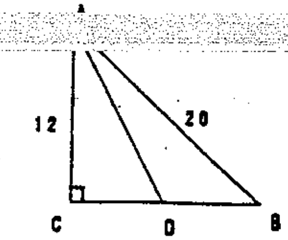


3. To swim a mile in a certain rectangular swimming pool, one must either swim the long length 80 times or negotiate the perimeter of the pool 22 times. What is the total number of square yards in the area of the pool? Hint: There are 1760 yards in one mile.

4. Determine the area of the right triangle ABC if $a + b = 17$.

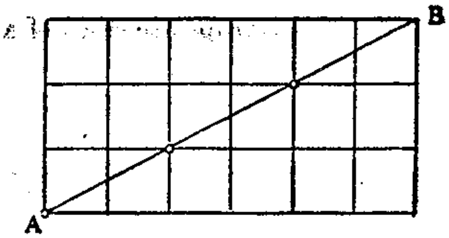


5. In right triangle ABC, if $AD = DB + 8$, find CD.



6. In the diagram, the diagonal AB passes through 6 squares of the 3 by 6 rectangle. As you see from the example, passing through a vertex does not count as passing through a square. Determine the number of squares that the diagonal AB passes through :

a) for a 3 by 4 rectangle
b) for a 4 by 6 rectangle
c) for a 12 by 18 rectangle



7. Find four consecutive positive integers such that the sum of the cubes of the smallest three is the cube of the fourth.

8. The coefficient of n^2 in the expansion of $n(n + 1)(n + 2)$ is 3.
a) What is the coefficient of n^3 in the expansion of $n(n +1)(n +2)(n +3)$?
b) What is the coefficient of n^4 in the expansion of $n(n +1)(n +2)(n +3)(n +4)$?

9. Factor the fourth degree polynomial $n^4 + 6n^3 + 11n^2 + 6n + 1$ as a product of two second degree poynomials.

10. Observe the following :

$$1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25 = 5^2$$
$$2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121 = 11^2$$
$$3 \cdot 4 \cdot 5 \cdot 6 + 1 = 361 = 19^2$$

a) Give the next two rows of the pattern.
b) Determine the value of m so that $n(n + 1)(n + 2)(n + 3) + 1 = m^2$.
(m should be expressed in terms of n.)

BRIEF SOLUTIONS

FIRST ROUND NOVEMBER 1995

1. a) $\frac{2}{1} \quad \frac{3}{3} \quad \frac{4}{6} \quad \frac{5}{10} \quad \frac{6}{15}$ b) $190 = 1 + 2 + 3 + ... + 19$

2. $57 = (7\cdot4) + (9\cdot5) - 2(4\cdot2)$

3. 396 yards. $80L = 1760$ and $(2L + 2W)(22) = 1760$ gives $L = 22$ and hence $W = 18$. Then $L\cdot W = 396$.

4. 30. If $a + b = 17$ then $a^2 + 2ab + b^2 = 289$. Since $a^2 + b^2 = 169$, subtracting gives $2ab = 120$. Dividing by 4 gives $\frac{1}{2}ab = 30$, the area of the triangle.

5. CD = 9. $20^2 = 12^2 + (CB)^2$ gives $CB = 16$. $(AD)^2 = 144 + (CD)^2$. Replacing AD by the given DB + 8 gives $(DB + 8)^2 = 144 + (16 - DB)^2$, and $DB = 7$. Then $CD = CB - DB = 16 - 7 = 9$.

6. a) 6 b) 8 c) 24 . For part (c), one way to count the squares is to stack up a bunch of 4 by 6 rectangles and use the count in part (b).

7. 3, 4, 5, 6. $(n-1)^3 + n^3 + (n+1)^3 = (n+2)^3$ simplifies to $(n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) = (n^3 + 6n^2 + 12n + 8)$ and then to $2n^3 - 6n^2 - 6n - 8 = 0$ and $n^3 - 3n^2 - 3n - 4 = 0$. Using synthetic (or long) division, $n = 4$ is the only real root and the four integers are 3, 4, 5, 6 .

8. a) 6 b) 10

9. $(n^2 + 3n + 1)(n^2 + 3n + 1)$. If you guess that the product looks like $(n^2 + an + 1)(n^2 + bn + 1)$ you can determine a and b by multiplying out and equating coefficients to the given 4^{th} degree polynomial.

10. a) $4\cdot5\cdot6\cdot7 + 1 = 29^2 = 841$ $5\cdot6\cdot7\cdot8 + 1 = 41^2 = 1681$
    b) $m = n^2 + 3n + 1$

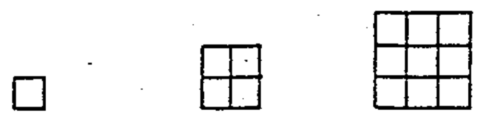
UNC MATHEMATICS CONTEST
FINAL ROUND
FEBRUARY 10, 1996

Take 3 hours for this exam. Your score will be determined by your work presented in the booklet. Calculators are permitted but are not necessary.

---

1. Find the smallest positive integer n such that the product (13)(19)(n) is also the product of three consecutive integers. [For example, the smallest positive integer n such that the product (3)(5)(n) is also the product of two consecutive integers is 2 since (3)(5)(2)=(5)(6).]

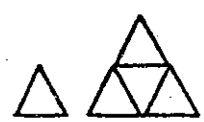
2. From the picture at the right you can see that it takes 4 unit length toothpicks to make a 1 by 1 square, 12 to make a subdivided 2 by 2 square and 24 to make a subdivided 3 by 3 square.



   (a) How many such toothpicks are needed to make a similar square pattern having side length 8?

   (b) Generalize to a side length of n.

3. It takes 3 toothpicks to make the small triangle and 9 to make the subdivided triangle having side length 2. How many are needed if the side length is:
   (a) 3? (b) 30? (c) n?

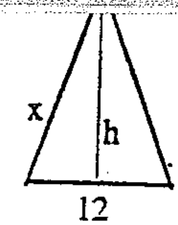


4. What is the smallest side length of a square such that a square and a triangle (these do not have the same side length) require the same number of toothpicks to make as described in problems 2 and 3?

5. The integer n is 124 less than one perfect square and 56 less than another perfect square. Determine n.

6. If all of the 720 permutations (or arrangements) of the digits 1, 2, 3, 4, 5, 6 are arranged in order of size when interpreted as six digit numbers, from the smallest 123456 to the largest 654321, what is the 423rd six digit number on that list? (As an example, the 6 permutations of 1, 2, 3 arranged in increasing order of size are 123, 132, 213, 231, 312, 321. The 4th permutation here is 231).

triangle having base length 12 such that its area and perimeter are numerically equal.



8. Observe the pattern to the right.

$$1^2=1$$
$$1^2-2^2+3^2=6$$
$$1^2-2^2+3^2-4^2+5^2=15$$

(a) What is the next line?

(b) Find a nice compact formula for

$1^2-2^2+3^2-4^2+...+n^2$ where n is an odd integer.

(As an example of a nice compact formula, the sum $1+3+5+...+2n-1$ can be expressed as n^2).

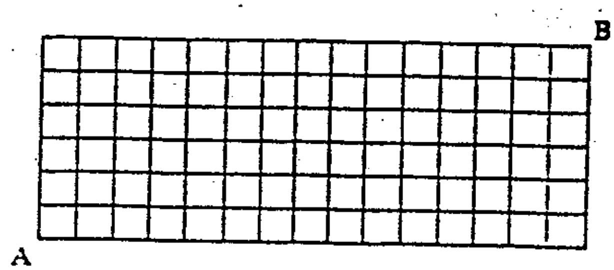
9. In the following sequence of square arrays

$$m(1) = [1] \quad , \quad m(2) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad , \quad m(3) = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix}$$

(a) what is the integer in the upper left hand corner of $m(10)$?

(b) the main diagonal of $m(3)$ is 6 10 14. Determine the sum of the elements on the main diagonal of $m(10)$.

10. (a) A 6 by 15 rectangle is subdivided into 90 one by one squares. Through how many squares does the diagonal AB pass? [Passing through a vertex does not count as passing through a square.]



(b) Through how many squares does the diagonal AB pass for an m by $2m$ rectangle that is subdivided into $2m \cdot m$ one by one squares?

(c) Through how many squares does the diagonal AB pass for an m by n rectangle that is subdivided into mn one by one squares? Explain.