University of Northern Colorado Statewide Mathematics Contest

First Round November 1996

- The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, ...
- An arithmetic progression is a sequence of numbers such that the difference between any two terms is constant.
- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b, c are the lengths of the appropriate sides of a right triangle.
- $1 + 2 + 3 + \ldots + n = n(n + 1)/2$

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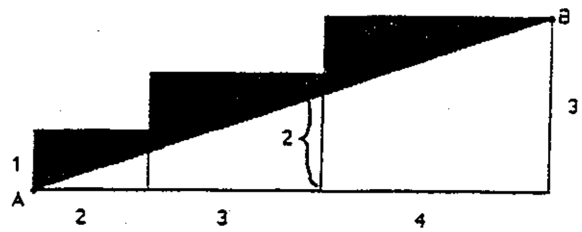
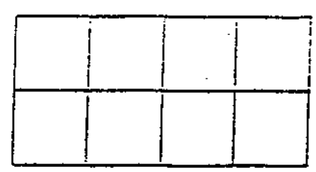
1. Determine the area of the shaded part formed by connecting points A and B; the rectangles are 1 by 2, 2 by 3, and 3 by 4.

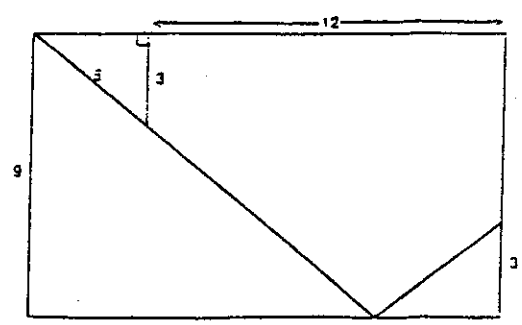


2. Express $(1 - \frac{2}{5})(1 - \frac{2}{7})(1 - \frac{2}{9})(1 - \frac{2}{11})(1 - \frac{2}{13}) \ldots (1 - \frac{2}{113})$ as a single fraction $\frac{a}{b}$.

3. The 2 by 4 rectangular grid requires 22 toothpicks to build. Determine the total number of toothpicks required to build a similar grid for each of the following sizes:
(A) 4 by 7
(B) 8 by 18



4. When the rectangle is cut into pieces and rearranged, a square can be formed. What is the perimeter of the square?



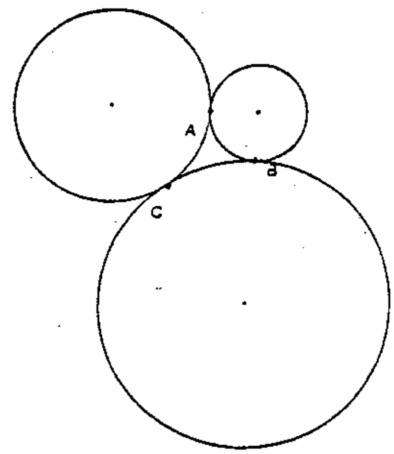
5. The sum of three consecutive integers in an arithmetic progression is 24. The sum of their squares is [illegible]. Determine the three integers.

6. The integer 5 can be written as a sum of three positive integers, where order matters, in the following six ways: $1+1+3$, $1+3+1$, $3+1+1$, $1+2+2$, $2+1+2$, $2+2+1$.

   (A) Determine how many ways there are of expressing 10 as a sum of three positive integers where order counts.

   (B) How many different such sequences of three positive integers have a sum of 50?

7. The three circles are mutually tangent externally as shown at points A, B, and C. If the radii of the circles are 1, 2, and 3, compute the length of chord AB.



8. Express $1+\frac{1}{2}(1+2)+\frac{1}{3}(1+2+3)+\frac{1}{4}(1+2+3+4)+\ldots+\frac{1}{20}(1+2+3+\ldots+20)$ as an integer.

9. Let r be a root of $x^5 - 1 = 0$ with $r \neq 1$. Compute the value of $r^{15} + r^{16} + r^{17} + \ldots + r^{49} + r^{50}$.

10. A pack of 7 cards numbered 1, 2, 3, 4, 5, 6, 7 is shuffled and then shuffled a second time in exactly the same manner, and then a third time in exactly the same manner, and so on. One such shuffle is represented by: $\begin{matrix}1\to2\\2\to3\\3\to1\end{matrix}\quad\begin{matrix}4\to5\\5\to4\end{matrix}\quad 6\to6\quad 7\to7$.

   A) How many shuffles of this type would it take to return these cards to their original order (1, 2, 3, 4, 5, 6, 7)?

   B) Another type of shuffle is given by $\begin{matrix}1\to2\\2\to1\end{matrix}\quad\begin{matrix}3\to4\\4\to5\\5\to6\\6\to3\end{matrix}\quad 7\to7$. How many shuffles of this type would it take to return these cards to their original order?

   C) What is the maximum number of shuffles, regardless of the type, required to have the 7 cards return to their original positions?

SOLUTIONS FIRST ROUND, NOVEMBER 1996

1. $\frac{13}{2}$; Shaded area = rectangle - triangle = $1 \times 2 + 2 \times 3 + 3 \times 4 - \frac{1}{2} \times 9 \times 3 = 20 - \frac{27}{2} = \frac{13}{2}$

2. 3/113; $\frac{3}{5} x \frac{5}{7} x \frac{7}{9} x \frac{9}{11} x \ldots x \frac{111}{113} = \frac{3}{113}$

3. (A) 5 x 7 + 8 x 4 = 67 (B) 8 x 19 + 18 x 9 = 314

4. 48; Area of rectangle is 16 x 9 = 144. The 12 x 12 square has perimeter 48.

5. -7, 8, 23; (a-d) + a + (a + d) = 24. Then a = 8. $(8-d)^2 + 8^2 + (8+d)^2 = 642$,
$64 - 16d + d^2 + 64 + 64 + 16d + d^2 = 642$; $450=2d^2$ d = 15 or d = -15.

6. (A) 36; 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36 (B) 1176; 1 + 2 + 3 + ... + 48 = 1176

7. $\sqrt{2}$; joining the centers of the circles forms a 3 - 4 - 5 right triangle. AB is then the hypotenuse of an isosceles right triangle.

8. 115; $\frac{2}{2}+\frac{1}{2}\frac{3x2}{2}+\frac{1}{3}\frac{4x3}{2}+\frac{1}{4}\frac{5x4}{2}+\ldots+\frac{1}{20}\frac{20x21}{2}=\frac{1}{2}(2+3+4+\ldots+21)=\frac{1}{2}\left(\frac{22\cdot 21}{2}-1\right)=115$

9. 1; $r^{15} + r^{16} + \ldots + r^{50} = r^{15}[1 + r + r^2 + \ldots + r^{35}] = r^{15}\left(\frac{r^{36}-1}{r-1}\right)=\left(\frac{r-1}{r-1}\right)=1$ ($r^{15}=1$ and $r^{36} = r$ since $r^5 = 1$).

10. (A) 6; *lcm*(3, 2, 1, 1) = 6 (B) 4; *lcm* (2, 4, 1) = 4
(C) 12; the partition of 7 that yields a maximum *lcm* is 7 = 3 + 4, and 3 x 4 = 12

UNIVERSITY OF NORTHERN COLORADO
MATHEMATICS CONTEST
FINAL ROUND - FEBRUARY 1997

- The positive integers are : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, ...
- The area of a triangle is (base)(height)/2
- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b, c are appropriate side lengths of a right triangle.
- $1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$

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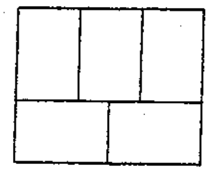
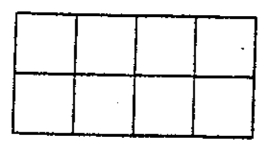
1. Five congruent rectangles are placed to form a larger rectangle with perimeter 44 inches. What is the area of the large rectangle?



2. An equilateral triangle and a regular hexagon (all six sides equal) have equal perimeter. The area of the triangle is 40 in^2. Determine the area of the hexagon.

3. A 2 by 4 rectangular grid requires 22 toothpicks to build. Determine the total number of toothpicks required to build a similar rectangular grid that has dimensions m by n.



4. (a) For which positive integers n is $\dfrac{8}{n-1}$ a positive integer?

   (b) What is the largest integer n for which $\dfrac{n^4 - 3n^3 - n^2 + 3n + 7}{n-3}$ is an integer?

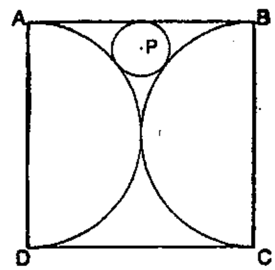
5. (a) The smallest positive integer with exactly eight positive divisors is 24 (include 1 and 24). The eight divisors are 1, 2, 3, 4, 6, 8, 12, 24. Find the next largest integer with exactly eight positive divisors.

   (b) The positive integer N has exactly twelve distinct positive divisors, but only three distinct prime divisors. If the sum of these three prime factors is 18, compute the smallest value of N.

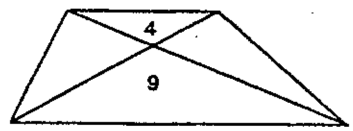
6. (a) Ten stools are arranged in a row. Three friends go in each day and always sit so that at least one stool is between two people. In how many ways can the stools be occupied, assuming the friends are indistinguishable (assume they were identical triplets)? For example, if there are just five stools, the following diagram shows that there is just one way of sitting: ⊗ ○ ⊗ ○ ⊗

(b) Generalize to *n* stools.

7. ABCD is a square. AD and BC are diameters of two tangent semicircles. Circle P is tangent to each semicircle and to AB. If DC = 18, what is the radius of circle P?



8. A trapezoid is divided into four triangles by its diagonals. Two of the areas are given. What is the total area of the trapezoid?



9. Let $x = 3^a + 3^b$ where a and b are chosen independently at random from the integers 1 through 100 inclusive (each integer has equal likelihood of being chosen). Compute the probability that x is a multiple of 5.

10. (a) Factor $x^8 + x^4 + 1$ as a product of a fourth degree polynomial and two second degree polynomials.

(b) Factor $x^{12} + x^8 + x^4 + 1$ completely so that the factors have integer coefficients.

(c) Generalize.

11. The integer 6 can be written as a sum of three positive integers where <u>order matters</u>, in the following ten ways: 1 + 1 + 4, 1 + 4 + 1, 4 + 1 + 1, 1 + 2 + 3, 1 + 3 + 2, 2 + 1 + 3, 2 + 3 + 1, 3 + 1 + 2, 3 + 2 + 1, 2 + 2 + 2

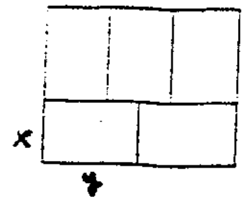
(a) Determine how many ways there are of expressing 20 as a sum of three positive integers, where order matters.

(b) Generalize and determine a formula for expressing *n* as such a sum of three positive integers.

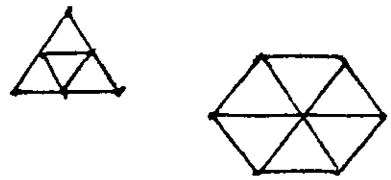
(c) Determine a formula for expressing *n* as such a sum of four positive integers, again where order matters.

1. 120 in^2 ; The perimeter is $5x + 4y = 44$. Also $3x = 2y$. Then $x = 4$, $y = 6$, and the area is

$12 \cdot 10 = 120$.



2. 60 in^2 ; From the pictures, each small triangle has area 10.



3. $2mn + m + n = n(m + 1) + m(n + 1)$

4. (a) 2, 3, 5, 9 (b) $n = 10$; group then divide. $\dfrac{n^3(n-3) - n(n-3) + 7}{n-3}$ has $\dfrac{7}{n-3}$ as a remainder. Now try various n.

5. (a) 30 ; $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdots$ has eight divisors and is smallest when a = 3, b = 1, c = 0, d = 0, ...

The next smallest occurs when a = 1, b = 1, c = 1, d = 0, ...

5. (b) 156 ; Examination of the primes 2, 3, 5, 7, 11, 13, ... shows two possibilities: 2, 3, 13 and 2, 5, 11; $2^2 \cdot 3 \cdot 13$ is smaller than $2^2 \cdot 5 \cdot 11$.

6. (a) 56 ; Draw diagram and enumerate the cases: (6 +5 +4 +3 +2 +1) + (5 +4 +3 +2 +1) + (4 +3 +2 +1) + (3 +2 +1) + (2 +1) + 1 = 56 = $\binom{8}{3}$.

6. (b) $\binom{n-2}{3}$; Two stools are forbidden; choose 3 of the remaining n - 2 stools.

7. 9/4 ; $(9 + r)^2 = 9^2 + (9 - r)^2$.