University of Northern Colorado
Mathematics Contest
FIRST ROUND
November 1997

The positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...
The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, ...
$1 + 2 + 3 + 4 + \ldots + n = n(n + 1)/2$
The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b, c are appropriate side lengths of a right triangle.

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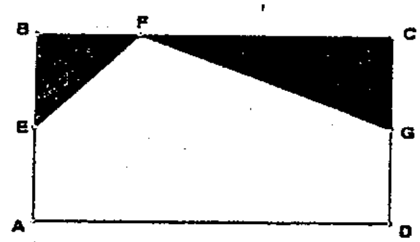
1. If you continue the pattern as shown, in which column would you find

   (a) 300? (b) 4231?

A	B	C	D	E
1	2	3	4	5
8	7	6		
9	10	11	12	13
16	15	14		
17	18	19	20	21
24	23	22		

2. ABCD is a rectangle with AD=14 and AB=6. E and G are midpoints of the sides and F is <u>any</u> point on $\overline{BC}$. What is the area of the shaded region?



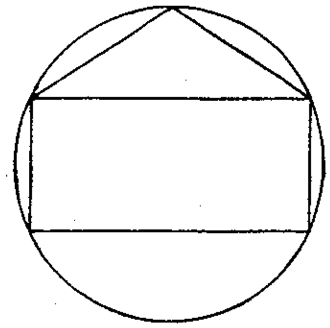
3. Express $\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{211}\right)$ as a fraction $\frac{a}{b}$.

4. In solving a quadratic equation Rodger misread the coefficient of the first-degree term and got 3 and –4 as solutions, while Richard misread the constant term and obtained –1 and 5 as solutions. Find the correct solutions.

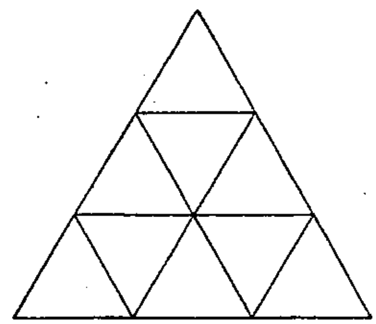
5. The length and width of a rectangle are prime numbers and the perimeter is 72. What is the maximum area possible?

6. Suppose you have 30 envelopes and you address 30 letters to match them. Closing your eyes you randomly stuff one letter into each envelope. In how many ways can you insert precisely 2 letters into the wrong envelopes and all others in the correct envelopes?

7. Inscribe a rectangle having base b and height h and an isosceles triangle of base b in a circle of radius 1 as shown in the figure. For what value of h do the rectangle and triangle have the same area?

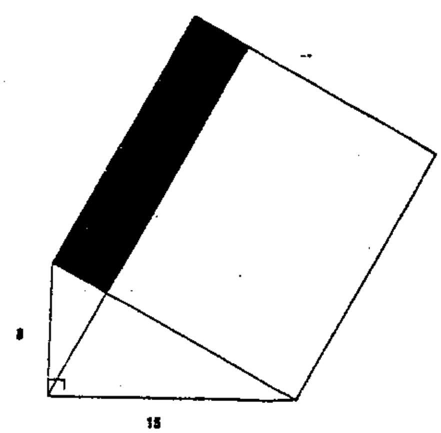


8. The equilateral triangle of side length 3 is subdivided into 9 smaller triangles. Place the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 in them so that the sum of all four numbers in any of the equilateral triangles with side length 2 is the same. What is the smallest value of this sum?



9. An integer is called decreasing if each digit is less than the one to its left. For example 6320 is a 4-digit decreasing number.
   a) How many 3 digit decreasing integers are there? That is, how many decreasing integers occur between 100 and 999?
   b) How many 4 digit decreasing integers are there?

10. A right triangle with side lengths 8 and 15 has a square sitting on its hypotenuse. What is the area of the shaded rectangle?



11. How many different fractions $\frac{m}{n}$ can you make if m and n are positive integers, $m < n$, $m + n = 575$, and also, each fraction is reduced to lowest terms?

BRIEF SOLUTIONS – First Round, November 1997

1. (a) D; all of the numbers in column D have a remainder of 4 upon division by 8.

   (b) B; the numbers in column B alternate with remainders of 2 or 7 upon division by 8.

2. 21 square units; since F is any point on $\overline{BC}$, let it be B. The area is ¼ the area of the rectangle.

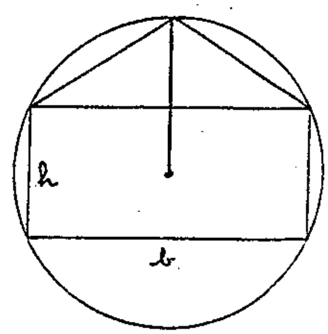
3. $\frac{1}{211}$; $\frac{1}{3}\cdot\frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdots\frac{209}{211}=\frac{1}{211}$.

4. 6 and –2; since $(x - 3)(x + 4) = x^2 + x-12$ and $(x + 1)(x – 5) = x^2 – 4x – 5$ the correct quadratic must be $x^2 – 4x – 12 = (x – 6)(x + 2)$.

5. 323; $2a + 2b = 72$ and $a + b = 36$. Checking among the primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 there are three pairs that sum to 36: 5 and 31, 7 and 29, 17 and 19. The largest area is 17 x 19 = 323

6. 435; select 2 in $\binom{30}{2}$ ways. These two must be interchanged. Alternatively label the envelopes 1, 2, ..., 30. 1 could be interchanged with any one of 2, 3, ..., or 30, or 2 could be interchanged with one of 3, 4, ..., 30, or ... This total is $29 + 28 + 27 + \ldots + 1 = \frac{29\cdot 30}{2} = 435$

7. $h=\frac{2}{5}$; setting the two areas equal gives
   $hb=\frac{1}{2}b(1-\frac{h}{2}))$, $2h = 1 -\frac{h}{2}$



8. 17; place the smallest numbers 1, 2, 3 in the three small triangles that overlap. The sum of the entries in the three 2 sided triangles is (1 + 2 + 3 + ... + 9 + 6)/3 = 17. Now, place the largest numbers at the corners so that intermediate sums 2 + 3 + 7, 1 + 2 + 8 and 1 + 3 + 9 differ by 1. Then fill in 4, 5, 6

9. (a) 120; there are $\binom{10}{3}=\frac{10\cdot 9\cdot 8}{6}=120$ of them.

(b) 210; there are $\binom{10}{4}=\frac{10\cdot 9\cdot 8\cdot 7}{24}=210$ of them.

Alternatively (for (a)) you could count these directly:

		total
between 100-199 there are none		0
between 200-299 there is just one 210		1
between 300-399	320, 321, and 310	3
between 400-499	432, 431, 430, 421, 420, 410	6

and so on. The total sum is $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 = 120$

10. 64 square units; by the Pythagorean Theorem the length of one side of the rectangle is 17. Let x be the small side. By similar triangles $\frac{x}{8}=\frac{8}{17}$. Then $x=\frac{64}{17}$ and the area of the shaded rectangle is $17\cdot\frac{64}{17}=64$.

11. 220; First, $575 = 5^2\cdot 23$. The list $\frac{1}{574},\frac{2}{573},\frac{3}{572},\ldots,\frac{287}{288}$ contains 287 fractions. We need to delete those fractions $\frac{a}{b}=\frac{a}{575-a}$ where a is a multiple of 5 and those where a is a multiple of 23 and then add back in those where $a = 115$ and $a = 230$, multiples of 5 and 23.
ANS $= 287 - 12 - 57 + 2 = 220$

University of Northern Colorado
Mathematics Contest
Final Round – February 1998

- The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, ...
- The positive perfect squares are 1, 4, 9, 16, 25, 36, 49, ...
- $1 + 2 + 3 + \dots + n = n(n+1)/2$

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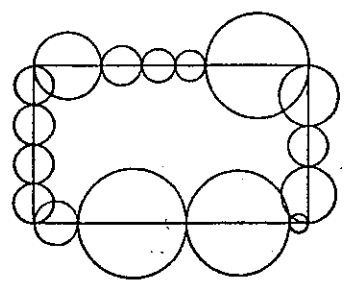
1. (a) Choose a number from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} at random. What is the probability that the number has an <u>odd</u> number of divisors? As an example, the number 6 has four divisors 1, 2, 3, and 6, counting 1 and the number itself.

   (b) Choose a number from {1, 2, 3, 4, ..., 100}. What is the probability that it has an odd number of divisors?

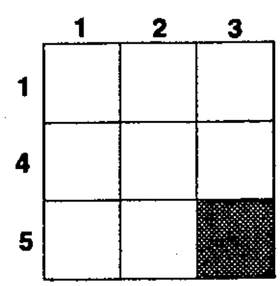
2. Give an example of a 4 digit perfect square having the form aabb. As an illustration, the number 6622 has this form, but is <u>not</u> a perfect square.

3. The rectangle in the picture has dimensions 4 by 7. Determine the sum of all the circumferences of all the circles as drawn. The center of each circle lies on the rectangle.



4. What is the sum of the roots of $(x-1)(x-2)(x-3)(x-4)-(x-2)(x-3)(x-4)(x-5) = 0$?

5. A "cross number" puzzle – Insert eight appropriate digits (one in each box, and they are not all different) to provide answers to the six questions:



ACROSS:	DOWN:
1. Square of a prime	1. Square of another prime
4. A prime number	2. A square
5. A square	3. A prime number

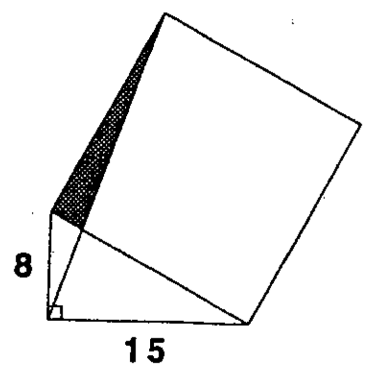
6. If you continue the pattern what is the
   (a) number at the beginning of the 29th row?
   (b) number at the beginning of the n-th row?
   (c) number at the end of the 51st row?
   (d) sum of the integers in the 20th row?
   (e) sum of the integers in the n-th row?

1	2					
3	4	5				
6	7	8	9			
10	11	12	13	14		
15	16	17	18	19	20	
21	22	23	24	25	26	27

...

7. Suppose you have 30 envelopes and 30 letters addressed to match them. Closing your eyes you randomly stuff one letter into each envelope. In how many ways can you insert precisely 3 letters into the wrong envelopes (and the other 27 into their correct envelopes)? Explain your reasoning.

8. A right triangle with side lengths 8 and 15 has a square sitting on its hypotenuse. What is the area of the shaded triangle?



9. How many different fractions $\frac{m}{n}$ can you make if m and n are positive integers, $m < n$, $m + n = 665$, and each fraction is reduced to lowest terms? (Hint: Even though $5 + 660 = 665$, $\frac{5}{660}$ is not allowed because it is not reduced.) Explain your reasoning.

10. A set of consecutive integers beginning with 1 is given. One of the numbers is erased and the arithmetic mean of the remaining numbers is 35 7/17. What number was erased?

BRIEF SOLUTIONS
Final Round
February 1998

1. (a) 3/10; only squares have an odd number of divisors.

   (b) 1/10

2. 7744; The form *aabb* is short for the representation $a10^3 + \text{a}10^2 + b \cdot 10 + b = a(1100) + b(11)$.

   Hence, *aabb* is divisible by 11. Checking squares of multiples of 11, namely $11 \cdot 1, 11 \cdot 2, 11 \cdot 3,$ $11 \cdot 4, 11 \cdot 5, 11 \cdot 6, 11 \cdot 7, 11 \cdot 8$, and $11 \cdot 9$ only 88^2 has the form *aabb*. Or, *aabb* is divisible by 121, so try perfect square multiples of 121, starting with 9.

3. 22π; the diameters of all circles add up to $4 + 7 + 4 + 7 = 22$.

4. 9; $(x\text{-}1)(x\text{-}2)(x\text{-}3)(x\text{-}4) - (x\text{-}2)(x\text{-}3)(x\text{-}4)(x\text{-}5) = (x\text{-}2)(x\text{-}3)(x\text{-}4)[(x\text{-}1) - (x - 5)] = 4(x\text{-}2)(x\text{-}3)(x\text{-}4)$;

   the roots are 2, 3, 4.

5. The first row is 169 (This number must be the square of one of 11, 13, 17, 19, 23, 29, 31); the second row is 277 and the third 16.

6. (a) $1 + 2 + 3 + \ldots + 29 = 435$ (b) $n(n + 1)12$, a triangular number (c) 1377; if you add 1 to each term in the sequence 2, 5, 9, 14, 20, … you get 3, 6, 10, 15, 21, … a shifting of the sequence of triangular numbers; these have general form $\frac{(n+1)(n+2)}{2}$. So the general form for the terms of the sequence 2, 5, 9, 14, 20, … is $\frac{(n+1)(n+2)}{2} - 1$.

   (d) $4620 = 210 + 211 + \ldots + 230$

   (e) $\frac{n(n+1(n+2)}{2}$; $\left[\frac{(n+1)(n+2)}{2} - 1\right]\left[\frac{(n+1)(n+2)}{2}\right] \div 2 - \left[\frac{n(n+1)}{2} - 1\right]\left[\frac{n(n+1)}{2}\right] \div 2 = n(n+1)(n+2)/2$;

   or examine the pattern of row sums 3, 12, 30, 60, 105 … Doubling gives 6, 24, 60, 120, 210, … and factoring gives 1·2·3, 2·3·4, 3·4·5, 4·5·6, 5·6·7, …

7. $2\binom{30}{3}$; First, choose which 3 are to be "deranged" in $\binom{30}{3}$ ways. Then, the 3 elements a, b, c can be deranged in exactly 2 ways; b, c, a and c, a, b.

8. $\frac{289(32)}{409}$; If you write the equation of the line through (0, -17) and (a, b) and set $y = 0$ you get the height of the triangle.

$$\frac{a}{8} = \frac{8}{17} \Rightarrow a = \frac{64}{17};\ \frac{b}{a} = \frac{15}{8} \Rightarrow b = \frac{15a}{8} = \frac{15}{8}\frac{64}{17} = \frac{120}{17}$$

$$y + 17 = \frac{\frac{120}{17} + 17}{\frac{64}{17}}x = \frac{120 + 289}{64}x\text{; when } y = 0,$$

$$x = \frac{17 \cdot 64}{409}.\quad Area = \frac{1}{2}(17)\frac{17(64)}{409} = \frac{289 \cdot 32}{409}$$

9. 216; $665 = 5 \cdot 7 \cdot 19$. There are 332 numbers in the set $\left\{\frac{1}{664}, \frac{2}{663}, \frac{3}{662}, \ldots, \frac{332}{333}\right\}$ since $\frac{a}{b} = \frac{1}{5 \cdot 7 \cdot 19}$ we need to remove multiples of a like 5a, $7a$, $19a$. But then we need to add back in multiples of a like $5 \cdot 7a$, $5 \cdot 19a$, $7 \cdot 19a$. Between 1 and 332 there are 66 multiples of 5, 47 multiples of 7, 17 multiples of 19, 9 multiples of 35, 3 multiples of 95 and 2 multiples of 133. The answer is then $332 - 66 - 47 - 17 + 9 + 3 + 2 = 216$.

10. 7 is erased; the average of 1, 2, 3, …, n is $\frac{n+1}{2}$. Delete n and the average drops to $\frac{n}{2}$. If we delete 1, the average increases to $\frac{n}{2} + 1$. Thus $\frac{n}{2} \le 35\,{}^{7}\!/_{17} \le \frac{n+2}{2}$ and $68\frac{14}{17} \le n \le 70\frac{14}{17}$. So $n = 69$ or 70. The average of the $n - 1$ numbers being $35\,{}^{7}\!/_{17}$ implies that $(n-1)\,35\,{}^{7}\!/_{17}$ must be an integer or that $n - 1$ is a multiple of 17, forcing $n = 69$. $1 + 2 + \ldots + 69 = 2415$, and $35\,{}^{7}\!/_{17} \times 68 = 2408$. The number erased is $2415 - 2408 = 7$.