University of Northern Colorado
Mathematics Contest
FIRST ROUND
November 1998

- The Pythagorean Theorem states that $c^2 = a^2 + b^2$ where a, b, c are appropriate side lengths of a right triangle.
- $1+2+3+4+\cdots+n = n(n+1)/2$

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1. Fill in the four missing numbers so that after the first two terms each number is the sum of the two before it:

   4, ____, ____, ____, ____, 47.

2. The fraction $\frac{1}{25}$ can be expressed in decimal form as .04, requiring two decimal places, fill in the chart giving the number of decimal places required in the decimal representation for each fraction :

Fraction	$\frac{1}{5}$	$\frac{1}{5^2}$	$\frac{1}{5^3}$	$\frac{1}{5^4}$	$\frac{1}{5^{1998}}$
Decimal places		2			

3. Select two different numbers from $\{1, 2, 3, 4, 5, 6\}$ at random. What is the probability that the sum of the two numbers is greater than their product? Express your answer as a fraction.

4. Determine all values of x so that $(x+3)^{(x-2)} = 1$. (Hint: When do you have $a^b = 1$?)

5. The numbers 1 through 16 can be placed in a 4 by 4 array so that row sums, column sums and the sum of the four corners are the same. Each number is used once. Determine the missing numbers.

		12	1
11			8
5		3	
4			15

6. Each student in a classroom shakes hands with everyone else exactly one time. How many students are there if there were a total of 91 handshakes?

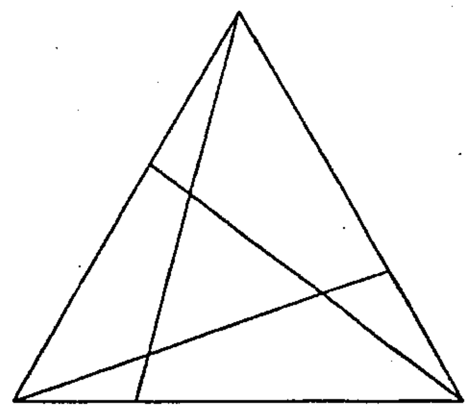
7. Determine the area of the triangle having vertices in the plane at the points $(2,3)$, $(17,11)$, $(17,3)$.

8. Determine all integers n such that $1+3^n+3^{2n} = 757$.

9. The sum of the lengths of the three sides of a right triangle is 90 in. The sum of the squares of the lengths of the three sides is 3362 sq. inches. Determine the area of the triangle.

10. A cevian of a triangle is a line drawn from a vertex to any point on the opposite side. Let $R(n)$ denote the maximum number of regions formed within the triangle by drawing n cevians from each vertex. The picture for the case $n=1$ is shown. Compute:

(a) $R(0)$ (b) $R(1)$ (c) $R(2)$ (d) $R(3)$



11. Consider the "integer triangle".
(a) How many terms are in the 50th row?
(b) What is the sum of the terms in the 50th row?

1	1st row
1 2 1	2nd row
1 2 3 2 1	3rd row
1 2 3 4 3 2 1	4th row
1 2 3 4 5 4 3 2 1	5th row
...	

BRIEF SOLUTIONS
FIRST ROUND – November 1998

1. 7, 11, 18, 29; Trial and error, or algebra.

2. 1, 2, 3, 4, 1998.

3. $\frac{10}{30} = \frac{1}{3}$

4. –2, 2, -4; $a^b = 1$ when $a = 1$, or $b = 0$ (and $a \neq 0$), or $a = -1$ and b is even.

5.

14	7	12	1
11	9	6	8
5	16	3	10
4	2	13	15

$1 + 2 + 3 + 4 + \ldots + 16 = 16(17)/2 = 136$
Each row and column sum is 136 / 4 = 34. This fact and trial and error finishes the table.

14	7	12	1
11	2	13	8
5	16	3	10
4	9	6	15

6. 14; Experiment with 2 students, then 3, 4, 5, ….

7. 60; Since the side lengths are 8, 15, 17 the triangle is a right triangle. Area = $\frac{1}{2} \cdot 15 \cdot 8 = 60$.

8. n = 3; Let $x = 3^n$, and solve $1 + x + x^2 = 757$. $x^2 + x - 756 = 0 = (x - 27)(x + 28)$. If x = 27, n = 3. x = -28 won't work.

9. 180; $a + b + c = 90$, $a^2 + b^2 + c^2 = 3362$, $a^2 + b^2 = c^2$, $a^2 + b^2 = 1681 = 41^2$, $c = 41$, $a + b = 49$, $a^2 + 2ab + b^2 = 2401$, $2ab = 720$, $\frac{1}{2}ab = 180$.

10. R(0) = 1, R(1) = 7, R(2) = 19, R(3) = 37.

11. (a). 99 (b). 2500

University of Northern Colorado
Mathematics Contest
Grades 7-12
FINAL ROUND – February 1999

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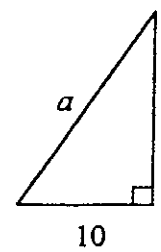
1. Certain fractions can be represented with terminating decimals. For example, the fraction ¼ can be expressed in the decimal form as .25, terminating after two decimal places. How many decimal places are in the decimal representation of:

   (a) $\frac{1}{2^n}$ (b) $\frac{1}{5^n}$ (c) $\frac{1}{2^m 5^n}$

2. A right triangle has side lengths a, b, and 10, with a being the length of the hypotenuse. Determine a and b if they are both integers.

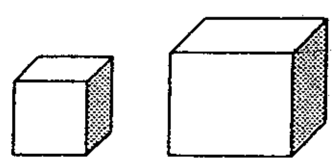


3. A proper fraction is one in which the numerator is less than the denominator. Determine the sum of all positive proper fractions (reduced and not reduced) whose denominators are less than or equal to 100. As an example, if you replace 100 by 4 the answer is:

$$\frac{1}{2}+\frac{1}{3}+\frac{2}{3}+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}=3$$

4. Fifty tickets numbered consecutively from 1 to 50 are placed in a jar. Two are drawn at random (without replacement). What is the probability that the difference of the two numbers drawn is 10 or less?

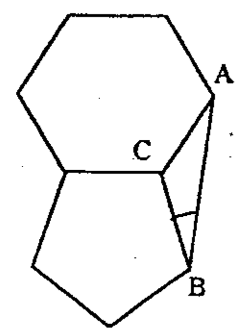
5. Two cubes have edges with integral lengths. The numerical value of the combined volume of the cubes is equal to the combined lengths of their edges. What are the measures of the side lengths of the two cubes?



6. $(x-2)^4-(x-2)=0$ and $x^2-kx+k=0$ have two roots in common. Determine k so that this is true.

7. Determine the area of the triangle having vertices at the three points (a, b), (a – 5, b + 12), and (a + 7, b + 17).

8. A regular n-gon has n sides of equal measure and all interior angles having equal measure. The figure shows a regular hexagon adjacent to a regular pentagon. Determine the measure of angle ABC.

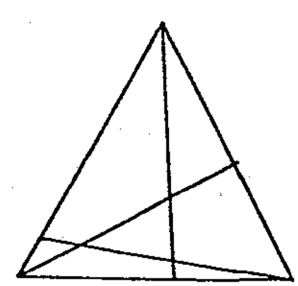


9. Determine all real values of x so that $(x^2-1)^{(x^2-9x+20)}=1$

10. Consider the "integer triangle". Give, with proof, a formula for the sum
(a) of the elements in the n^{th} row.
(b) of all the elements through the n^{th} row.
(c) Determine other properties of this triangle.

				1					Row 1
			1	3	1				Row 2
		1	3	5	3	1			Row 3
	1	3	5	7	5	3	1		Row 4
1	3	5	7	9	7	5	3	1	Row 5
				. . .					

11. A cevian of a triangle is a line drawn from a vertex to the opposite side. Let R(n) denote the maximum number of regions formed within a triangle by drawing n cevians from each vertex. The picture for the case n=1 is shown. Determine, with proof, a formula for R(n). Generalize.



Solutions for the FINAL ROUND
FEBRUARY 6, 1999

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1. (a) n (b) n (c) max (m,n)

2. a = 26, b = 24; Recalling that 5, 12, 13, is a Pythagorean triplet, double all sides. Or try trial and error on $a^2 - b^2 = 100$.

3. 2475; Use $1 + 2 + \ldots + n = n(n+1)/2$ many times.

4. 89 / 245; There are $\binom{50}{2}$ or 1225 ways of selecting two. 40 pairs differ by 10, namely, {50, 40}, {49, 39}, …, {12, 2}, {11, 1}. 41 pairs differ by 9, 42 by 8, …, 49 by 1. The probability is $(40 + 41 + \ldots + 49) \div 1225 = 445 \div 1225$.

5. 2 and 4.; $a^3 + b^3 = 12a + 12b$, $a^2 - ab + b^2 = 12$. Use the quadratic formula on $a^2 - ab + (b^2 - 12) = 0$ to get $a = \frac{b \pm \sqrt{(16 - b^2)3}}{2}$. Now try b = 1, 2, 3, 4, … and only b = 2 works. Trial and error on the original equation also works.

6. k = 3; Use $x^3 - 1 = (x-1)(x^2 + x + 1)$ to obtain $(x-2)^3 - 1 = (x-3)[(x-2)^2 + (x-2) + 1]$ $= (x-3)(x^2 - 3x + 3)$. Alternatively, the roots of the quartic are 2, 3, and $\frac{3}{2} \pm \frac{1}{2}\sqrt{3}\,i$. The roots of $x^2 - kx + k = 0$ are $\frac{k \pm \sqrt{k^2 - 4k}}{2}$, so pick k = 3.

7. 84.5; Translate so that (a, b) = (0, 0). The triangle in question has side lengths 13, 13, $13\sqrt{2}$ and is, therefore, a right triangle.

8. 24°; $\angle ACB = 360° - 120° - 108° = 132°$. Then $\angle ABC = \frac{1}{2}(180° - 132°) = 24°$.

9. $0, -\sqrt{2}, +\sqrt{2}, 4, 5$; $a^b = 1$ when a = 1, or when b = 0 and a $\neq$ 0, or when a = −1 and b is even. When $x^2 - 1 = 1$, $x = \pm\sqrt{2}$; when $x^2 - 9x + 20 = 0$, x = 4, 5 (and $x^2 - 1 \neq 0$); when $x^2 - 1 = -1$, x = 0 (and $x^2 - 9x + 20$ is even).

10. (a) $2n^2 - 2n + 1$; Use $1 + 3 + 5 + \ldots + 2n - 1 = n^2$ repeatedly. The row sums are 1, 4 + 1, 9 + 4, 16 + 9, 25 + 16, 25 +16, … or $n^2 + (n - 1)^2 = 2n^2 - 2n + 1$.

(b) $\frac{n}{3}(2n^2 + 1)$; The total sum is $(1^2 + 2^2 + \ldots + n^2) + (1^2 + 2^2 + \ldots + (n - 1)^2)$ which can be simplified. Or use $k^2 = \binom{k}{2} + \binom{k+1}{2}$.

Then $\sum_{K=1}^{n} k^2 + \sum_{K=1}^{n-1} k^2 = \binom{n+1}{3} + \binom{n+2}{3} + \binom{n}{3} + \binom{n+1}{3} = 4\binom{n+1}{3} + n$.

11. $R(n) = 3n^2 + 3n + 1$; There are various proofs.