Mathematics Contest

FIRST ROUND

For Colorado Students Grades 7 - 12

November 6, 1999

- The 10 digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- The average of n numbers is their sum divided by n.
- A triangle is isosceles if it has at least two congruent sides.

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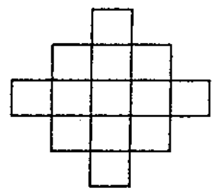
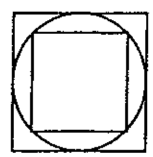
1. A classroom has a rectangular array of desks. A student notices that there are three desks to her right, two to her left, five desks in front of her and two behind her. How many desks are in the room?

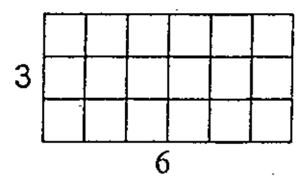
2. The figure is made up of congruent square tiles. If the total area is 52 square inches, what is the perimeter of this "cross like" figure?



3. A square is inscribed in a circle which is inscribed in a square. Determine the ratio of the area of the small square to the large square.



4. The figure shows a 3 by 6 rectangle divided into unit squares. There are a total of 32 squares of all sizes in this figure: 18 1 by 1, 10 2 by 2, and 4 3 by 3 squares. If a 3 by n rectangle has a <u>total</u> of 152 squares, what is n?

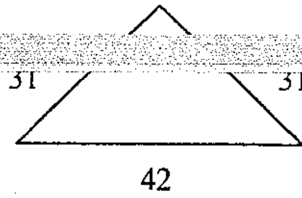


5. If $m + n = 5$ and $mn = 2$, determine the value of:

   (a) $m^2 + n^2$ (b) $m^3 + n^3$

OVER

make if each side is a whole number (integer) length and the perimeter is 104 inches? One possible triangle is drawn.



7. What number should be removed from the list 1, 2, 3, 4, . . . , 21 so that the average of the remaining numbers is $10\frac{3}{4}$?

8. The 4 by 4 square shown has two diagonals: 1, 6, 11, 16 and 4, 7, 10, 13, each with the same sum 34. This square is made by filling in each unit square with consecutive numbers 1, 2, 3, . . ., beginning in the upper-left hand corner and ending in the lower right-hand corner. Give the sum of the diagonals in a similar:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(a) 2 by 2 square (b) 3 by 3 square

(c) 5 by 5 square (d) 10 by 10 square

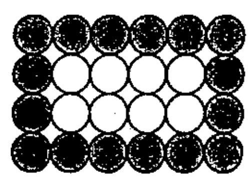
9. A three-digit number *abc* is a "peak number" if *b*, the middle digit, is strictly bigger than *a* and strictly bigger than *c*. For example, 290 and 131 are "peak numbers" but 432 and 288 are <u>not</u> .

(a) What is the largest three-digit "peak number"?

(b) What is the smallest three-digit "peak number"?

(c) How many three-digit "peak numbers" are there?

10. The picture shown is made by making a rectangle of white disks and then creating a border of black disks around the rectangle. Give the dimensions of the outside rectangle for <u>all</u> such configurations such that the number of black disks equals the number of white disks. (The picture shows a configuration with a 4 by 6 outside rectangle that does <u>not</u> work since it uses 8 white disks and 16 black disks)



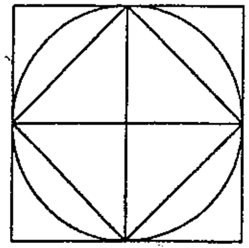
11. The polynomial equation $x^2 - 2 = 0$, with integer coefficients, has $\sqrt{2}$ as a root, since $\sqrt{2}$ satisfies the equation $x^2 - 2 = 0$. What polynomial equation with integer coefficients has $\sqrt{2} + \sqrt{3}$ as a root?

NOVEMBER 9, 1999

1. 48; the dimensions are 6 by 8.

2. 40; each small square has area 4, side length 2.

3. ½ ; rotate the square 45° and draw in the two diagonals = diameters.



4. 26; $3n + 2(n-1) + (n-2) = 6n - 4 = 152; 6n = 156, n = 26$.

5. (a) 21; $(m+n)^2 = m^2 + 2mn + n^2 \Rightarrow m^2 + n^2 = (m+n)^2 - 2mn = 25 - 4 = 21$.

(b) 95; $(m+n)^3 = m^3 + 3m^2n + 3mn^2 + n^3 \Rightarrow m^3 + n^3 = (m+n)^3 - 3mn(m+n)$

$= 125 - 6 \cdot 5 = 95$.

6. 25; The possible lengths of the "other" side are 2, 4, 6, 8, …, 50. So there are 25 such isosceles triangles.

7. 16; Let n denote the number to be removed. $1 + 2 + \ldots + 21 = 22(21)/2 = 231$. $(231 - n)/20 = 43/4$.

Now solve for n.

8. (a) 5; (b) 15; (c) 65; (d) 505

9. (a) 898; (b) 120; (c) 240; count by letting the middle number range from 2 to 9.

10. 5 by 12 or 6 by 8; let the outside dimensions be a and b. Then $2a + 2b - 4 = (a-2)(b-2)$

$2a + 2b - 4 = ab - 2a - 2b + 4$, $4a + 4b = ab + 8$, $b = \dfrac{8-4a}{4-a} = \dfrac{4a-8}{a-4} = \dfrac{4a-16+8}{a-4} = 4 + \dfrac{8}{a-4}$.

Now try $a = 5, 6, 8, 12$.

11. $x^4 - 10x^2 + 1 = 0$; $x = \sqrt{2} + \sqrt{3}$, $x - \sqrt{2} = \sqrt{3}$, $x^2 - 2\sqrt{2}x + 2 = 3$, $x^2 + 1 = 2\sqrt{2}x$,

$x^4 - 2x^2 + 1 = 8x^2$, $x^4 - 10x^2 + 1 = 0$

University of Northern Colorado

Grades 7 – 12

FINAL ROUND - February 5, 2000

- $1 + 2 + 3 + \ldots + n = n(n+1)/2$
- The binomial coefficient $\binom{n}{k}$ gives the number of ways of choosing k things out of n things.

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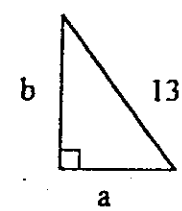
1. Determine all positive integers a and b such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)=\frac{3}{2}$$

2. The right triangle has hypotenuse of length 13 and legs a and b. If the area of the triangle is 14, what is the sum a + b?



3. Consider the positive integers that have all of their digits restricted to the set $\{2, 3, 4, 5, 6, 7, 8\}$. In their natural order these integers form the unending sequence

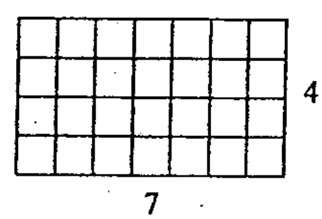
$$2, 3, 4, 5, 6, 7, 8, 22, 23, 24, 25, 26, 27, 28, 32, 33, \ldots$$

(a) What number follows 888 ?

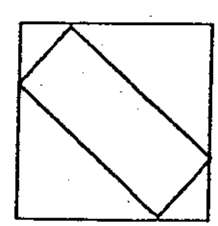
(b) What is the 2817th number in this sequence?

4. A 4 by n rectangle divided into unit squares contains 240 squares of all sizes. What is n ?

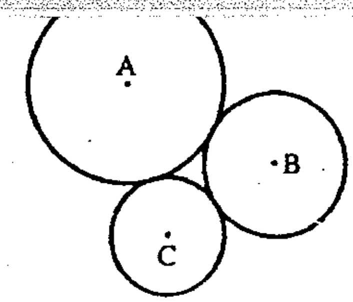
   In the figure shown, the 4 by 7 rectangle contains 60 squares of all sizes. There are 28 1 by 1 squares, 18 2 by 2 squares, 10 3 by 3 squares, and 4 4 by 4 squares for a total of 60 squares.



5. Inscribe a rectangle in a square so that each of the four triangles formed is isosceles. Determine the length of the diagonal of the rectangle if the sum of the areas of these four triangles is 200 square inches.



a, b, and c, respectively. The lengths of segments AB, BC and CA are 23, 17 and 12, respectively. Determine the three radii a, b and c.



7. The 5 by 5 square has two diagonals 1, 7, 13, 19, 25 and 5, 9, 13, 17, 21. Each has the same sum: $1 + 7 + 13 + 19 + 25 = 65$.

   (a) What is the sum of the elements of one diagonal in a 20 by 20 square?

   (b) What is the sum in an n by n square?

   (c) Describe other "diagonals" in an n by n square with this common sum.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

8. Let $S(n) = \{1, 2, 3, 4, \ldots, n\}$. (a) For which n can you split up or separate this set into two disjoint sets so that the sums of the elements in each set are the same? Verify your results. For example, $S(4) = \{1, 2, 3, 4\}$ can be split into $\{1, 4\}$ and $\{2, 3\}$, with $1 + 4 = 2 + 3 = 5$. But $S(5) = \{1, 2, 3, 4, 5\}$ <u>cannot</u> be split this way.

   (b) Generalize or extend this result to the case of <u>three</u> disjoint sets.

9. In the xy- plane what is the length of the shortest path from the origin (0, 0) to the point (12, 16) that does not pass through the circle $(x-6)^2 + (y-8)^2 = 25$? You can travel along the circumference, but do not pierce the circle.

10. The polynomial equation $x^4 - 10x^2 + 1 = 0$, with <u>integer coefficients</u>, has $\sqrt{2} + \sqrt{3}$ as a root. Determine a polynomial equation with <u>integer coefficients</u> that has $\sqrt[4]{3+\sqrt{8}} + \sqrt[4]{3-\sqrt{8}}$ as a root.

11. A five-digit is a "peak number" if the first three digits are in strict <u>ascending</u> order and the last three are in strict <u>descending</u> order. The middle digit is the peak. For example, 24840 is a "peak number" but 14722 and 32752 are not.

    (a) Determine the number of five digit "peak numbers".

    (b) Determine the number of five-digit "peak numbers" that do <u>not</u> contain zeros. (Try to express your answer in terms of binomial coefficients).

    (c) Extend this problem to seven-digit numbers.

1. $\{3, 8\}$ and $\{4, 5\}$; Clearing fractions gives $2(a+1)(b+1) = 3ab$,

$2(ab+a+b+1) = 3ab$, $2a+2b+2 = ab$; Now solve for b as

$b = \frac{2a+2}{a-2} = \frac{2a-4+6}{a-2} = 2\frac{(a-2)}{a-2} + \frac{6}{a-2} = 2 + \frac{6}{a-2}$. Now try values of a, keeping in mind that, a and b are positive integers.

2. 15; $a^2 + b^2 = 169$; $\frac{1}{2}ab = 14$ is the area. $(a+b)^2 = a^2 + 2ab + b^2$ gives $(a+b)^2 = 169 + 56 = 225$ and then $a + b = 15$.

3. (a) 2222 (b) 22244; The number 8888 is the $7 + 7^2 + 7^3 + 7^4 = 2800$th number.

4. 25; $240 = 4n + 3(n-1) + 2(n-2) + (n-3)$. These are $4n$ 1 by 1, $3(n-1)$ 2 by 2, $2(n-2)$ 3 by 3 and $n-3$ 4 by 4 squares.

5. 20; Call the diagonal d. With appropriate labels, $d^2 = a^2 + b^2$, (here a, b represent the hypotenuse of each isosceles triangle). $200 = \text{AREA} = \left[\frac{1}{2}x^2 + \frac{1}{2}y^2\right] = x^2 + y^2$

But $2x^2 = b^2$ and $2y^2 = a^2$, so $400 = 2x^2 + 2y^2 = a^2 + b^2 = d^2$.

Another way: Specialize the rectangle to be a square. The diagonal is now the same length as the side of the outside square. This outside square is now seen to be made up of 8 isosceles right triangles, each having areas 50. Total area = 400; side = diagonal = 20.

6. $a = 9$, $b = 14$, $c = 3$; $a + b = 23$, $b + c = 17$, $a + c = 12$.
Now add: $2a + 2b + 2c = 52$, giving $a + b + c = 26$. Subtracting $a + b = 23$ gives $c = 3$ and so on.

7. (a) 4010 (b) $n(n^2+1)/2 = 1 + (n+2) + (2n+3) + (3n+4) + \ldots + [(n-1)n + n]$

(c) Generalized diagonals, consisting of an integer one from each row and column has this common sum.

8. (a) $n = 4k$ or $4k - 1$. Since the sum must be even, look at even triangular numbers. A triangular number has the form $T_n = n(n+1)/2$ and is even whenever n is a multiple of 4 or one less than a multiple of 4.

(b) $n = 6k$ or $6k - 1$

9. $10\sqrt{3} + \frac{5}{[illegible]} \approx 22.5565$. Draw this carefully on a piece of graph paper. Travel from (10,10) along a tangent line to the circle, then along an arc of the circle that subtends a 60° angle, and then along another tangent line.

10. $x^4 - 4x^2 - 4 = 0$. Let $a = \sqrt[4]{3 + \sqrt{8}}$ and $b = \sqrt[4]{3 - \sqrt{8}}$. Then $a^4 + b^4 = 6$, $ab = \sqrt[4]{(3+\sqrt{8})(3-\sqrt{8})} = 1$,

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = 6 + 6 + 4ab(a^2 + b^2)$$

$(a+b)^4 = 12 + 4(a^2 + b^2)$. Now express $a^2 + b^2$ in terms of the root $a + b$. $(a+b)^2 = a^2 + 2ab + b^2$

so that $a^2 + b^2 = (a+b)^2 - 2ab$.

Finally, $(a+b)^4 = 12 + 4\left((a+b)^2 - 2\right) = 12 + 4(a+b)^2 - 8$ or $(a+b)^4 - 4(a+b)^2 - 4 = 0$. The equation becomes $x^4 - 4x^2 - 4 = 0$.

11.

(a) $\binom{2}{2}\binom{3}{2} + \binom{3}{2}\binom{4}{2} + \binom{4}{2}\binom{5}{2} + \ldots + \binom{8}{2}\binom{9}{2} = 1 \cdot 3 + 3 \cdot 6 + 6 \cdot 10 + 10 \cdot 15 + 15 \cdot 21 + 21 \cdot 28$

$+ 28 \cdot 36 = 2142$

(b) $\binom{2}{2}^2 + \binom{3}{2}^2 + \binom{4}{2}^2 + \binom{5}{2}^2 + \binom{6}{2}^2 + \binom{7}{2}^2 + \binom{8}{2}^2 =$

$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + 21^2 + 28^2 = 1596$

(c) Allowing zeros: $\binom{3}{3}\binom{4}{3} + \binom{4}{3}\binom{5}{3} + \binom{5}{3}\binom{6}{3} + \ldots + \binom{8}{3}\binom{9}{3}$

Not allowing zeros: $\binom{3}{3}^2 + \binom{4}{3}^2 + \binom{5}{3}^2 + \ldots + \binom{8}{3}^2$

For part (b), for example, you look at cases. If 9 is the middle number choose 2 of 1, 2, 3, 4, 5,6, 7, 8 in $\binom{8}{2}$ ways to go before 9 and after the 9. Total in this case: $\binom{8}{2}^2$. If 8 is the middle number, there are $\binom{7}{2}^2$ ways to complete. This combinatorial argument extends.