Problems for Theory of Equations – Math 520 – Summer 2002

1. When is $x^n - c^n$ divisible by $x - c$?
2. When is $x^n + c^n$ divisible by $x + c$?
3. Without actually dividing, show that $x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$ is divisible by $x^2 + x + 1$.
4. Show that $(x+1)^n - x^n - 1$ is divisible by $x^2 + x + 1$ only if n is an odd number not divisible by 3.
5. The equation $f(x) = x^n - nx + n - 1 = 0$, $n > 1$, is satisfied by $x = 1$. What is the multiplicity of this root?
6. Show that $(x - 1)^2$ is a factor of $x^n - nx + n - 1$ when $n > 1$.
7. Points representing the roots of the equation $3x^3 + 4x^2 + 8x + 24 = 0$ are on a circle centered at (0, 0). Find these roots.
8. The equation $2x^4 + x^3 - 2x - 8 = 0$ has four distinct roots of equal moduli. Find them.
9. Let p(x) be a polynomial with integer coefficients. Prove that if both p(0) and p(1) are odd numbers, then $p(x) = 0$ cannot have integer roots.
10. Show by substitution that if $a + bi$ is a root of the equation $x^3 + px^2 + qx + r = 0$, and if p, q, r are real, then $a - bi$ is a root of the equation.
11. Find the equation of lowest degree satisfied by the primitive 40^{th} roots of unity.
12. Determine p and q so that $x^4 - 9x^3 + px^2 + qx + 25 = 0$ has 5 as a double root.
13. If r and s are roots of $x^2 - 50x + 498 = 0$ determine the value of $r^2 + rs + s^2$.
14. Factor $x^5 + x + 1$ in Z[x].
15. If a, b, c are different numbers such that

    $a^3 + 3a + 14 = 0$
    $b^3 + 3b + 14 = 0$
    $c^3 + 3c + 14 = 0$

    determine the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

16. Determine the four values of n in the set {2, 3, 4, …, 15} such that $(x + 1)^n - (x^n + 1)$ is divisible by $x^2 + x + 1$.

17. A regular decagon $P_1 P_2 \ldots P_{10}$ is drawn in the coordinate plane with P_1 at (1, 0) and P_6 at (3, 0). If P_n is the point (x_n, y_n) compute the numerical value of the product $(x_1 + y_1 i)(x_2 + y_2 i) \ldots (x_{10} + y_{10} i)$.

18. Find the real root of:

    (a) $(1 + i)x^3 + (1 + 2i)x^2 - (1 + 4i)x - 1 + i = 0$

    (b) $(1 + i)x^3 + (1 + 2i)x^2 - (1 + i)x - 1 - 2i = 0$

19. Determine the two square roots of

    (a) i (b) –i

20. Solve the quadratic equation: $(2 - 2i)x^2 - (11 + 9i)x - 16 + 16i = 0$.

21. Solve the equation

    (a) $x^3 = -8i$ (b) $x^4 + 4 = 0$ [graph your result]

22. Explain why $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ satisfies the equation $x^2 + x + 1 = 0$.

23. Show why $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ satisfies the $x^4 + x^3 + x^2 + x + 1 = 0$.

24. Use a TI-92 or TI-89 to investigate when $x^2 + x + 1$ divides $x^{2n} + x^n + 1$. Prove your theorem.

25. A polynomial p(x) has a remainder of 3 when divided by x – 1 and a remainder of 5 when divided by x – 3. What is the remainder when p(x) is divided by $(x - 1)(x - 3)$?

26. Let $V_1, V_2, V_3, \ldots, V_n$ be n equally spaced points on a unit circle and let P be any point on the circle. Conjecture and prove a formula for the sum $\overline{PV_1}^2 + \overline{PV_2}^2 + \ldots + \overline{PV_n}^2$.

27. Let $a_1, a_2, a_3, \ldots, a_n$ denote the roots of $x^n - a_1^n = 0$. Show that $(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) = n\,a_1^{n-1}$.

28. Let $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$. Show that $x^7 - x = x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$; recall that you add and multiply modulo 7.

29. Express tan(nx) in terms of tanx for n = 1, 2, 3, 4, 5. Generalize and try to prove your generalization.

30. Verify that $\frac{\sqrt{5}-1}{4} = \frac{1}{\sqrt[5]{179+80\sqrt{5}}}$. Each is an expression for $\sin 18^\circ$, showing that $\sin 18^\circ$ is an irrational number.

31. If p, q, and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, what is the value of $p^3 + q^3 + r^3$?

32. Given three vertices $2 + i$, $1 - 3i$, $-2 + i$ of a parallelogram, determine the fourth vertex.

33. Let a, b, c be the roots of $x^3 + 3x + 3 = 0$. Determine the value of $(a + b)(b + 1)(c + 1)$.