THE DISCRIMINANT

Let $f(x) = a_0(x - x_1)(x - x_2)...(x - x_n) = a_0x^n + a_1x^{n-1} + ... + a_n$

Definition: The discriminant of f(x) is the expression

$D = a_0^{2n-2}(x_1 - x_2)^2 \cdots (x_1 - x_n)^2(x_2 - x_3)^2 ... (x_2 - x_n)^2 ... (x_{n-1} - x_n)^2$

This product of n(n – 1)/2 differences $(x_i - x_j)$ squared is a symmetric function, and can be expressed as a polynomial in the elementary symmetric functions $\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0} ... \frac{a_n}{a_0}$.

TASK 1 Write out D for f(x) = (x –a)(x – b).

TASK 2 Write out D for f(x) = (x – a)(x – b)(x – c).

TASK 3 Write out D for $f(x) = a_0(x - a)(x - b)(x - c)(x - d)$.

TASK 4 When does the discriminant vanish?

TASK 5 The discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$. Show that this is consistent with the present more general definition of D by doing the following:

(a) Let $ax^2 + bx + c = a(x - r_1)(x - r_2)$ and express $r_1^2 + r_2^2$ in terms of a, b, c.

(b) Express $D = (r_1 - r_2)^2$ in terms of a, b, c.

TASK 6 What can you say about the polynomial f(x) if $D > 0$?

TASK 7 What can you say if $D < 0$?

TASK 8 What is the purpose of the factor a_0^{2n-2} ?

The discriminant of $ax^3 + bx^2 + cx + d = 0$ is
$D = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$

TASK 1 Find D for $x^3+3x^2- 2 - 5 = 0$ and describe the roots.

TASK 2 Find D for $x^3 + px + q = 0$.

TASK 3 Show that $x^3 + px + 16 = 0$ has three real roots if $p \leq -12$.

TASK 4 Describe in words what you would have to do to show that $D = -4p^3 - 27q^2$ for $x^3 + px + q = 0$ directly (i.e., using the definition that $D = (a - b)(a - c)(b - c)$, where a, b, c are the roots.

TASK 5 Earlier you showed that if $a_1, a_2, \ldots, a_n$ are the roots of $x^n - a_1^n = 0$ then $(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) = na_1^{n-1}$. Use this to find D for each of the following:

(a) $x^3 - 1 = 0$

(b) $x^4 - 1 = 0$

(c) $x^5 - 1 = 0$

(d) $x^n - 1 = 0$