EVEN AND ODD FUNCTIONS

Definitions Let D be a set of numbers such that –r is in D whenever r is in D. If $f(-x) = f(x)$ for all x in D, then f is an even function on D. If $f(-x) = -f(x)$ for all x in D, then f is an odd function on D.

Example 1 Let $F(x) = x^3 - (1/x)$ and $G(x) = (\sin x)/(x + x^3)$ on the set D of all real numbers except zero. Then one sees from

$$F(-x) = (-x)^3 - 1/(-x) = -x^3 + (1/x) = -F(x),$$

$$G(-x)\frac{\sin(-x)}{(-x)+(-x)^3} = \frac{-\sin x}{-\left(x+x^3\right)} = \frac{\sin x}{x+x^3} = G(x),$$

that F is an odd function and G is an even function on D.

Example 2 Let $H(x) = \cos x + \sin x$ on the set R of all real numbers. Then $H(-x) = \cos(-x) + \sin(-x) = \cos(x) - \sin(x)$. Since $H(-x)$ does not appear to be equal to $H(x)$ or $-H(x)$ for all x in R, we suspect that H is neither even nor odd on R. To prove that H is not even on R it is sufficient to find one value of x in R for which $H(-x) \neq H(x)$; to prove H not odd on T one needs only to find one x in R with $H(-x) \neq -H(x)$.

Trying $x = \pi/4$, one sees that

$$H\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$H\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) = 0$$

Since $H(-\pi/4)$ is neither $H(\pi/4)$ nor $-H(\pi/4)$, this one choice of x shows that H is neither even nor odd on R. (If one tried $x = \pi/2$ instead of $x = \pi/4$, the results would not be helpful. Why not?)

<u>PROBLEMS</u>

1. Let H(x) be defined on the set R of all reall numbers, let $H(\pi/2) = 1$, and $H(-\pi/2) = -1$
   (a) Can H be an even function on the set R? Explain.
   (b) Is H necessarily an odd function on R? Explain.
   (Hint: See Example 2)
2. Let f(2) = 3 and f(-2) = -7. Also let 2 and -2 be in D.
   (a) Can f be an even function on D? Explain.
   (b) Can f be an odd function in D? Explain.
3. On the domain R of all real numbers, let f(x) = sin(x), g(x) = cos(x), and $h(x) = x + x^2$. Show that one of these functions is even on R, one is odd, and one is neither even nor odd.
4. Let $F(x) = 2^x$, $G(x) = 2^x + 2^{-x}$, and $H(x) = 2^x - 2^{-x}$.
   (a) Graph y = F(x), y = G(x), and y = H(x) for $-3 \leq x \leq 3$.
   (b) Which of these graphs is symmetric with respect to the y-axis, which is symmetric with respect to the origin, and which one has neither of the symmetries?
   (c) Which of F, G, and H is an even function on [-3, 3], which is odd, and which is neither even nor odd?
5. For each of the following formulas for f(x), determine whether f is even, odd, or neither even nor odd on the set R of real numbers:

   (a) x^2; (b) $4x^3 - 2x$; (c) $3x - 5$; (d) $\dfrac{x}{5x^4 + 7}$; (e) $|x|$;

   (f) $\sqrt[3]{x^5 - 2x}$; (g) x sin(x); (h) $x^3\cos(x)$
6. Do as in Problem 5 for the following:

   (a) sin(x); (b) $\sqrt[3]{x^4 + 3}$; (c) $\sin^2 x$; (d) $\dfrac{\sin x}{2} + \cos x$;

   (e) $x + |x|$; (f) x + cos(x)
7. Let the polynomial P be defined on the set R of all real numbers by $P(x) = ax^3 + bx^2 + cx + d$. What must be true of the numbers a, b, c, and d for P to be an even function on R? For P to be an odd function on R?

8. Let $F(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + ax + a_0$. What must be true of the real numbers a_i for the polynomial F to be even on the set R of all real numbers? For F to be odd on R?

9. Let f be defined on the closed interval; [-1, 1], i.e., for $-1 \leq x \leq 1$.

   Let $g(x) = f(x) + f(-x)$ and $h(x) = f(x) - f(-x)$.

   (a) Is the function g even, odd, or neither on [-1, 1]? Explain.

   (b) Is the function h even, odd, or neither on [-1, 1]? Explain.

10. Let f be an odd function on a set D and let 0 be in D. Show that $f(0) = 0$.

11. Determine all functions f that are both even and odd on [-1, 1].

12. Show that every function F on [-a,a] can be expressed as G + H with G even and H odd on [-a, a].

13. Let I be the open interval $-\pi/2 < x < \pi/2$. For which positive integers n is:

    (a) $x^n(\sin x + \tan x)$ even on I?

    (b) $x^n(\sin x + \tan x)$ odd on I?

    (c) $x^n \sec x$ even on I?

    (d) $x^n \sec x$ odd on I?

14. Let f and F be even and let g and G be odd on [-a, a]. Which of the following are even functions and which are odd functions on [-a, a]?

    (a) fF ; (b) FG ; (c) gG ; (d) f^7g^6 ; (e) f^6g^7 ;

    (f) f^7g^7 ; (g) $|fg|$; (h) $f + F$; (i) $f - F$; (j) $g + G$;

    (k) $g - G$